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Analysis of *Varroosis* Model in Honeybee Colony with Interventions

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Abstract

A deterministic mathematical model is proposed and analyzed to study the transmission dynamics of *Varroosis* in honeybee colony with interventions. The study combined both treatment and biocontrol strategy on curbing the menace of *Varroosis* on honeybee colony. As such, the study established the existence of the most important four steady states that include: disease-free and infestation-free, infestation with virus-free *Varroa-mite*s, infestation with virus-carrying *Varroa-mite*s and endemic steady state. Moreover, the study established the existence of backward bifurcation and sensitivity analysis of the model was performed. Correspondingly, the analysis of the model reveals that, ineffective treatment can induce backward bifurcation. Furthermore, the study results indicated that, when treatment is 100% effective, the disease-free and infestation-free steady state is globally asymptotically stable for $R_0 < 1$, whereas for $R_0 > 1$the global stability of the endemic steady state is proved only on a special case.

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1. Introduction

Honeybee has been substantially influencing modern agricultural production through pollination processes in sustaining ecosystem. As such, the majority of food crops consumed by human depend largely on honeybees' pollination [1]. In addition, the generous contributions of honeybees could not be over-emphasizing in producing honey, beeswax, and other products which are used for vast purposes.

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Honeybee population is confronted increasingly with enormous challenges extending from diseases, predators, parasitic *Varroa-mite*, viruses, brood diseases, pesticides, inadequate nutrition, climate & seasonal changes and the stresses of moving colonies for crop pollination [2-4].

Varroa-mite (*Varroa-destructor*) is an ectoparasite that attacks honeybee colony. It infests and transmits pathogens (viruses) via vertically or horizontally within and between colonies. In a nutshell, *Varroa-mite* feeds on honeybee body, reproducing and surviving on brood cells and suppressing honeybee immunity which eventually results to death. Moreover, viral infection spreads via feeding process when honeybee contacted with contaminated utensils or a *Varroa-mite* carrying virus feeds on an uninfected honeybee, it might release the virus into the honeybee's haemolymph and a virus-free *Varroa-mite* feeds on an already infected honeybee, it can acquire the virus [5-8].

Interventions such as biological, chemical, mechanical and heating methods have long been put to use to curb the menace of *Varroosis* as studied by [9-11]. Furthermore, mathematical models were developed and studied on the dynamics of *Varroa-mite* infestation and vectored disease transmission into honeybee population. Also, [12] examined honeybee-mite interactions in the presence of migration effects on their population dynamics. A mathematical model studied and analyzed on honeybee, *Varroa-mite*, and Acute honeybee paralysis virus interactions with seasonality [13]. Bifurcation analysis is one of focus of this study which prompted to reviewing [14], whose study investigated the existence of bifurcation on the transmission dynamics of HIV/AIDS on CD count. Additionally, a study proposed a mathematical model of *Varroa-mite* and the Acute Bee Paralysis Virus (ABPV), in which the honeybee population is divided into hive bees and forager bees based on tasks performed in the colony [15]. In short, gargantuan models were developed on honeybee such includes: [16-19] and among others.

This study is committed to extending the model developed due to [11] by remedying the model shortfalls. Their model didn't consider *Varroa-mite* population (virus-free *Varroa-mite* and *Varroa-mite* carrying virus). Similarly, control strategies using treatment and biocontrol agent (*Pseudoscorpions*) for *Varroosis* were not addressed. Hence, this study incooperated these components in order to have a model mimicking real life phenomenon and reducing the impact of *Varroosis* on honeybee colony. It also aimed to investigate existence of bifurcation, global stability and sensitivity analysis of the model.

2. Methodology

2.1. Model formulation

The total population$N(t)$of honeybee, at time t, is divided into six (6) mutually exclusive compartments, namely: Susceptible honeybee $S(t)$, honeybee infested by virus-free *Varroa-mite* $M(t)$, honeybee infested by *Varroa-mite* carrying virus $I(t)$, Population of virus-free *Varroa-mite* $V_f(t)$, Population of *Varroa-mite* carrying virus$V_v(t)$and Population of biocontrol agent $A_B(t)$. Thus, $N(t) = S(t) + M(t) + I(t)$

It is assumed that infested and infected honeybee has (N_m andN_i) a shorter life span than healthy beesN_s. Also, assuming that *Varroa-mites* have higher interaction with broods and hives than forager and that rate of disinfestation in honeybee depends on its state of health, hence;$\alpha_1 > \alpha_2$. It is assumed that the biocontrol agent feeds on both virus-free and virus-carrying *varro-mite*, the population growth for biocontrol agent is assumed to be logistic, the recruitment rate of brood population is assumed to be constant, the population growth for *Varroa-mite* is assumed to be logistic and the carrying capacity for the mite changes with host (honeybee) population size and natural death rate for all the populations is assumed to be constant. Infection induced death rate for all the populations is assumed to be constant. Also, it is only a horizontal transmission mode is assumed and *Varroa-mites* carrying virus is constantly recruited into the colony.

2.2. Model description

The model describes the interaction between honeybee population, *Varroa-mite* population and population of biocontrol agent in a single honeybee colony. The model provides control strategies that minimize the effects of both infestation and infection. The population of susceptible honeybee $S(t)$is naturally recruited at the rate$A$, disinfestation rate $\alpha_1$andα_2, treatment rate$\pi$from both$M(t)$ and $I(t)$respectively. It decreases by transmission rate of infestation by virus-free *Varroa-mite*β_1. The population of honeybee infested by virus-free *Varroa-mite* $M(t)$ grows by the rate of infestation by virus-free *Varroa-mite* $\beta_1$and decline by rate of infestation by *Varroa-mite* carrying virusβ_2, disinfestation

Table 1. State Variables

Symbol	Descriptions
$S(t)$	Susceptible honeybee at time t
$M(t)$	Honeybee infested by virus-free *Varroa-mite* at time t
$I(t)$	Honeybee infested by *Varroa-mite* carrying virus at time t
$V_f(t)$	Population of virus-free *Varroa-mite* at time t
$V_v(t)$	Population of *Varroa-mite* carrying virus at time t
$A_B(t)$	Population of biocontrol agent at time t
N_s	Total subpopulation of healthy honeybee
N_m	Total subpopulation of honeybee infested by virus-free *Varroa-mite*
N_i	Total subpopulation of honeybee infected by *Varroa-mite* carrying virus
N	Total population of honeybee

Table 2. Parameters

Symbol	Descriptions
β_1	transmission rate of infestation by virus-free *Varroa-mite*
β_2	transmission rate of infestation by virus-carrying *Varroa-mite*
α_1	Disinfestation rate for honeybees infested by virus-free *Varroa-mite*
α_2	Disinfestation rate for honeybees infested by virus-carrying *Varroa-mite*
d	Natural death rate for all the populations of honeybees
δ	Infection induced death rate for all the populations of honeybees
τ	Intrinsic growth rate of biocontrol agent
γ	Intrinsic growth rate of *Varroa-mite*
k	Environmental carrying capacity for biocontrol agent
Q	Environmental carrying capacity for *Varroa-mite*
η_1	Rate at which virus-free *Varroa-mite* acquires virus
η_2	Rate at which virus-carrying *Varroa-mite* loss it to the host healthy bees
η_3	Constant recruitment rate of *Varroa-mite* carrying virus into the colony
C_1	Conversation coefficient of virus-free *Varroa-mite* to bio-agent
C_2	Conversation coefficient of virus-carrying *Varroa-mite* to bio-agent
μ_v	Natural death rate for the populations of *Varroa-mite*
μ	Natural death rate for the population of biocontrol agent
π	Treatment using Thymol powder
A	Recruitment rate of healthy bees in the colony

at the rateα_1, treatment at rateπ and natural death rate$d$respectively. The population of honeybee infested by *Varroa-mite* carrying virus $I(t)$ grows when *Varroa-mite* carrying virus infested on both susceptible honeybee and honeybee infested by virus-free *Varroa-mite* at the rateβ_1 and$\beta_2$respectively. It declines by disinfestation at rateα_2, treatment at rateπ, natural death rate$d$and infection induced death δ respectively. The population of virus-free *Varroa-mite* V_f naturally grows at the rate $\gamma\left(1-\frac{V_f}{Q(N)}\right)$ and get decline by conversion coefficient of biocontrol agent and its natural death at rate $C_1$and $\mu_v$respectively. The population of *Varroa-mite* carrying virus V_v naturally grows by acquiring the virus at rate η_1, grow by constant recruitment into a colony at $\eta_3$and get decline by losing the virus to healthy honeybee at η_2, conversion coefficient of biocontrol agent and its natural death at rate $C_2$and $\mu_v$respectively. The population of biocontrol agent (pseudoscorpion) naturally grows at rate $\tau\left(1-\frac{A_B}{k}\right)$ and conversion coefficients at rate C_1andC_2respectively. It declines by natural death at rateμ.

2.3. State variables and parameters

The state variables and parameters of the model are presented in Table 1 and Table 2 respectively:

2.4. Model equations

$$\dot{S}(t) = A - \beta_1 S\left(V_f + V_v + M + I\right) + (\alpha_1 + \pi) M + (\alpha_2 + \pi) I - dS \tag{1}$$

$$\dot{M}(t) = \beta_1 S\left(V_f + M\right) - \beta_2 M (V_v + I) - (\alpha_1 + \pi + d) M \tag{2}$$

$$\dot{I}(t) = \beta_1 S (V_v + I) + \beta_2 M (V_v + I) - (\alpha_2 + \pi + d + \delta) I \tag{3}$$

$$\dot{V}_f(t) = \gamma V_f\left(1 - \frac{V_f}{Q(N)}\right) - (C_1 A_B + \mu_v) V_f \tag{4}$$

$$\dot{V}_v(t) = \eta_1 V_f \frac{N_i}{N} - \eta_2 V_v \frac{N_s}{N} + \eta_3 - (C_2 A_B + \mu_v) V_v \tag{5}$$

$$\dot{A}_B(t) = \tau A_B\left(1 - \frac{A_B}{k}\right) + \left(C_1 V_f + C_2 V_v\right) A_B - \mu A_B \tag{6}$$

With initial conditions$S(0) \geq 0$, $M(0) \geq 0$,$I(0) \geq 0$, $V_f(0) \geq 0$, $V_v(0) \geq 0$ and $A_B(0) \geq 0$

3. Model Analysis

3.1. Existence of the steady states of the model

The steady states of the model are established, when the RHS of the equation (1)-(6) is set to zero and eventually solved.

3.1.1. Disease-free and infestation-free steady state

The model has a unique disease-free and infestation-free steady state given as

$$E_1^*(S^*, 0, 0, 0, 0, 0) = E_1^*\left(\frac{A}{d}, 0, 0, 0, 0, 0\right) \tag{7}$$

3.1.2. Infestation with virus-free Varroa-mites steady state

In this steady state, susceptible honeybees are infested with virus-free *Varroa-mite* only. Then,

$$I = V_v = A_B = 0 \tag{8}$$

Substituting equation (8) in to the system (1)-(6), we have:

$S^* = \frac{A_1 + \sqrt{A_2}}{2\beta_1 \gamma dN}$, $M^* = \frac{B_1 + \sqrt{A_2}}{2\beta_1 \gamma dN}$ and $V_f^* = \frac{Q(\gamma - \mu_v)}{\gamma N}$

For positive steady state, $\gamma > \mu_v$ holds.

Where:

$$A_1 = A\beta_1\gamma N + \gamma d^2 N + \alpha_1 \gamma dN + \pi\gamma dN + \beta_1 \gamma dQ - \beta_1 d\mu_v Q$$

$$A_2 = \left(A\beta_1\gamma N + \gamma d^2 N + (\alpha_1 + \pi)\gamma dN + \beta_1 dQ(\gamma - \mu_v)\right)^2 - 4A\beta_1\gamma^2 dN^2(\alpha_1 + d + \pi)$$

$$B_1 = A\beta_1\gamma N - \gamma d^2 N + \alpha_1 \gamma d(-N) - \pi\gamma dN - \beta_1 \gamma dQ + \beta_1 d\mu_v Q$$

Thus, we obtain the infestation with virus-free *Varroa-mite* steady state given as

$$E_2^*\left(S^*, M^*, 0, V_f^*, 0, 0\right) = E_2^*\left(\frac{A_1 + \sqrt{A_2}}{2\beta_1\gamma dN}, \frac{B_1 + \sqrt{A_2}}{2\beta_1\gamma dN}, 0, \frac{Q(\gamma - \mu_v)}{\gamma N}, 0, 0\right) \tag{9}$$

3.1.3. Infestation with virus-carrying Varroa-mites steady state

In this steady state, susceptible honeybees are infested with virus-carrying *Varroa-mites* only. Then,

$$M = V_f = A_B = 0 \tag{10}$$

Substituting equation (10) in to the system of equation (1)-(6), we have:

$S^* = \frac{A_1+\sqrt{A_2}}{2\beta d(\mu_v N+\eta_2 N_s)}$, $I^* = -\frac{B_1+\sqrt{A_2}}{2\beta(d+\delta)(\mu_v N+\eta_2 N_s)}$ and $V_v^* = \frac{N\eta_3}{\mu_v N+\eta_2 N_2}$

Where:

$$A_1 = (\mu_v N + \eta_2 N_s)(A\beta + d(\alpha_2 + d + \delta + \pi)) + \beta N\eta_3(d+\delta),$$

$$A_2 = 2\beta N\eta_3(d+\delta)(\mu_v N + \eta_2 N_s)(A\beta + d(\alpha_2 + d + \delta + \pi)) + (\mu_v N + \eta_2 N_s)^2(A\beta - d(\alpha_2 + d + \delta + \pi))^2 + \beta^2 N^2 {\eta_3}^2 (d+\delta)^2 B_1$$

$$= (\mu_v N + \eta_2 N_s)(d(\alpha_2 + d + \delta + \pi) - A\beta) + \beta N\eta_3(d+\delta)$$

Thus, we obtain the infestation with virus-carrying *Varroa-mite* steady state given as:

$$E_3^*(S^*, 0, I^*, 0, V_v^*, 0) = E_3^* \begin{pmatrix} \frac{A_1+\sqrt{A_2}}{2\beta d(\mu_v N+\eta_2 N_s)}, 0, -\frac{B_1+\sqrt{A_2}}{2\beta(d+\delta)(\mu_v N+\eta_2 N_s)}, 0, \frac{N\eta_3}{\mu_v N+\eta_2 N_2}, 0 \\ 0 \end{pmatrix} \tag{11}$$

3.1.4. Existence of the endemic steady state and backward bifurcation

In this steady state, there exist all the population of honeybee, virus-free *Varroa-mite*, virus-carrying *Varroa-mite* and biocontrol agent.

Let $E_8^* = \left(S^*, M^*, I^*, V_f^*, V_v^*, A_B^*\right)$be any arbitrary steady state of the model (1)-(6). Further, let

$$\lambda^* = \beta_1 S^* \left(V_f^* + V_v^* + M^* + I^*\right) = \beta_1 S^* \left(V_f^* + M^*\right) + \beta_1 S^* \left(V_v^* + I^*\right) \tag{12}$$

Equation (12) is associated with forces of infestation and infection at steady state. Setting the right hand side of the model (1)-(6) to zero and solve for the state variables (non-zero solutions). We have:

$$\left.\begin{aligned} S^* &= \tfrac{1}{dw_3}\left(A_1 + \lambda A_{10} + A_5\right)^2 - A_7) \\ M^* &= \tfrac{1}{2\beta_2 w_2}\left(A_{15} + A_{14}\right)^2 \\ I^* &= \tfrac{1}{w_3}\left(A_{19} + \lambda\tfrac{1}{2\beta_2}A_{18}\right)^2 \\ V_f^* &= \tfrac{1}{C_1^2 kNQ-\gamma r}\left(A_{20} + \sqrt{A_{21}}\right) \\ V_v^* &= \tfrac{1}{2\gamma C_2^2 k}\left(A_{22} + \sqrt{A_{21}}\right) \\ A_B^* &= \tfrac{1}{C_1^2 kNQ-\gamma r}\left(A_{23} + \tfrac{\sqrt{A_{21}}}{2C_2}\right) \end{aligned}\right\} \tag{13}$$

Substituting (13) into the force of infestation and infection defined in (12), so that the non-zero (endemic) steady states of the model (12) satisfy equation (14). In addition, from equation (14) we obtained $\lambda^* = 0$as one of the solutions (that corresponds to disease-free and infestation-free steady state) and the core quartic polynomial corresponds to endemic steady state.

$$\lambda^*\left(a_1\lambda^{*4} + a_2\lambda^{*3} + a_3\lambda^{*2} + a_4\lambda^* + a_5\right) = 0 \tag{14}$$

Where:

$$a_1 = \frac{A_{10}^2 D_2}{w_2}, a_2 = \frac{A_{10}^2}{2\beta_2 w_2}, a_3 = \frac{A_{10}^2 D_7 D_6 D_2^2}{w_3}, a_4 = \frac{D_6}{2\beta_2 w_3}, a_5 = D_7 D_6, D_1 = A_1 + A_5, D_2 = A_{19} + \frac{1}{2\beta_2 w_2}A_{18},$$

$$D_3 = \left(A_{15} + A_{14}\right)^2, D_4 = \left(A_{20} + \sqrt{A_{21}}\right), D_5 = \left(A_{22} + \sqrt{A_{21}}\right), D_6 = \frac{\beta_1}{dw_3} + D_1^2 - A_7, D_7 = \frac{D_2^2 + D_3 + D_4 + D_5}{2d\beta_2 w_3 \gamma C_2^2 k\left(C_1^2 kNQ - \gamma r\right)},$$

$$A_1 = -Aw_3 - \frac{rw_1w_3}{4C_2} + \frac{\mu w_1w_3}{4C_2} - \frac{C_1NQw_1w_3}{4C_2} + \frac{rw_0w_3^2}{4C_2w_2} - \frac{\mu w_0w_3^2}{4C_2w_2} + \frac{C_1NQw_0w_3^2}{4C_2w_2} - \frac{C_1NQw_1w_3w_4}{4\gamma C_2}$$
$$+ \frac{C_1NQw_0w_3^2w_4}{4\gamma C_2w_2} - \frac{rw_1w_3w_5}{4C_2^2k} - \frac{C_1^2NQw_1w_3w_5}{4\gamma C_2^2} + \frac{rw_0w_3^2w_5}{4C_2^2kw_2} + \frac{C_1^2NQw_0w_3^2w_5}{4\gamma C_2^2w_2} + \frac{rw_0w_3^2\mu_v}{4C_2^2kw_2} - \frac{rw_1w_3\mu_v}{4C_2^2k}$$
$$- \frac{w_1w_3}{4\gamma C_2^2k} + \frac{C_1^2NQw_0w_3^2\mu_v}{4\gamma C_2^2w_2} - \frac{C_1^2NQw_1w_3\mu_v}{4\gamma C_2^2} + \frac{C_1NQw_1w_3\mu_v}{4\gamma C_2} - \frac{C_1NQw_0w_3^2\mu_v}{4\gamma C_2w_2} - \frac{w_0w_3^2}{2\beta_2} + \frac{w_1w_2w_3}{2\beta_2}$$

$$A_2 = \left(w_3 - \frac{w_1}{2} - \frac{w_0w_3}{2w_2}\right),$$

$$A_3 = 4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3r\right)$$
$$+\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2 + \frac{w_0w_3^2}{4\gamma C_2^2kw_2}$$

$$A_4 = 4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3r\right)$$
$$+\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2$$

$$A_5 = \frac{1}{2\beta_2}w_1\sqrt{} - 4\beta_2w_2w_3$$
$$+\left(\begin{array}{l} -w_2w_3 - 2\beta_2\beta_2 + \frac{1}{2\gamma C_2^2k}w_3\beta_2\left(\gamma C_2kr - \gamma C_2k\mu + \gamma C_1C_2kNQ + C_1C_2kNQw_4 + \gamma rw_5 + C_1^2kNQw_5 + \gamma r\mu_v\right) \\ +C_1^2kNQ\mu_v - C_1C_2kNQ\mu_v \\ +\sqrt{\left(\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2\right.} \\ \left.4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3r\right)\right) \end{array}\right)$$

$$A_6 = \left(-2\beta_2 + \frac{A_8}{2\gamma C_2^2k} - w_2w_3\right)^2 - 4\beta_2w_2w_3$$
$$A_7 = \frac{w_0w_3}{2\beta_2w_2}\sqrt{A_6}$$

$$A_8 = \beta_2w_3\left(-\gamma C_2k\mu + \gamma C_2C_1kNQ + \sqrt{A_9} + C_1^2kNQ\mu_v - C_2C_1kNQ\mu_v + C_1^2kNQw_5 + C_2C_1kNQw_4 + \gamma C_2kr + \gamma r\mu_v + \gamma rw_5\right)$$
$$A_9 = 4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3r\right)$$
$$+\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2$$

$$A_{10} = A_2\sqrt{A_3}\sqrt{A_4}$$

$$A_{11} = w_2w_3 + 2\beta_2 - \frac{1}{2\gamma C_2^2k}w_3\beta_2\left(\gamma C_2kr - \gamma C_2k\mu + \gamma C_1C_2kNQ + C_1C_2kNQw_4 + \gamma rw_5 + C_1^2kNQw_5 + \gamma r\mu_v\right.$$
$$\left.+C_1^2kNQ\mu_v - C_1C_2kNQ\mu_v+\right.$$

$$A_{12} = 4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3r\right)$$
$$+\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2$$

$$A_{13} = -C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 + \gamma C_2C_1Q - \gamma r\mu_v - \gamma rw_5 \\ +4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3 r\right)$$

$$A_{14} = -w_2w_3 - 2\beta_2 + \frac{1}{2\gamma C_2^2k}w_3\beta_2\left(\begin{array}{l}\gamma C_2kr - \gamma C_2k\mu + \gamma C_1C_2kNQ + C_1C_2kNQw_4 + \gamma rw_5 + C_1^2kNQw_5 + \gamma r\mu_v \\ +C_1^2kNQ\mu_v - C_1C_2kNQ\mu_v + \sqrt{}((\gamma C_2k\mu - \gamma C_2kr - kN + A_{18}\end{array}\right)$$

$$A_{15} = \left(A_{11} + \sqrt{A_{12}}\right) + \sqrt{}\left(-4\beta_2w_2w_3 + (A_{13})^2\right)$$

$$A_{16} = -\frac{rw_3\mu_v}{4C_2^2k} - \frac{rw_3w_5}{4C_2^2k} - \frac{w_3}{4\gamma C_2^2k} - \frac{C_1^2NQw_3\mu_v}{4\gamma C_2^2} + \frac{C_1NQw_3\mu_v}{4\gamma C_2} - \frac{C_1^2NQw_3w_5}{4\gamma C_2^2} - \frac{C_1NQw_3w_4}{4\gamma C_2} - \frac{C_1NQw_3}{4C_2} - \frac{rw_3}{4C_2} \\ +\frac{\mu w_3}{4C_2} - \frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{w_2w_3}{2\beta_2}$$

$$A_{17} = 4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3 r\right) \\ +\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2$$

$$A_{18} = \sqrt{}\left(-4\beta_2w_2w_3 + \left(-w_2w_3 - 2\beta_2 + \frac{1}{2\gamma C_2^2k}w_3\beta_2\left(\begin{array}{l}\gamma C_2kr - \gamma C_2k\mu + \gamma C_1C_2kNQ + C_1C_2kNQw_4 + \gamma rw_5 \\ +C_1^2kNQw_5 + \gamma r\mu_v + C_1^2kNQ\mu_v - C_1C_2kNQ\mu_v\end{array}\right)\right. \\ \left.+\sqrt{}\left(\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2\right.\right. \\ \left.\left.+4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3 r\right)\right)$$

$$A_{19} = A_{16}\sqrt{A_{17}}$$

$$A_{20} = \gamma(-N)Qr + \frac{1}{2}C_1kNQr - \frac{1}{2}C_1k\mu NQ - \frac{1}{2}C_1^2kN^2Q^2 - \frac{C_1^2kN^2Q^2w_4}{2\gamma} - \frac{C_1NQrw_5}{2C_2} - \frac{C_1^3kN^2Q^2\mu_v}{2\gamma C_2} + \frac{C_1^2kN^2Q^2\mu_v}{2\gamma} \\ +\frac{C_1^3kN^2Q^2w_5}{2\gamma C_2} - \frac{CNQ}{2\gamma C_2} - \frac{C_1NQr\mu_v}{2C_2} + NQr\mu_v$$

$$A_{21} = 4\gamma C_2^2k\left(C_1^2\eta_3kNQ - C_1kNQrw_4 + C_1k\mu NQw_4 - NQrw_4\mu_v + \gamma NQrw_4 + \gamma\eta_3 r\right) \\ +\left(\gamma C_2k\mu - \gamma C_2C_1kNQ - C_1^2kNQ\mu_v + C_2C_1kNQ\mu_v + C_1^2(-k)NQw_5 - C_2C_1kNQw_4 - \gamma C_2kr - \gamma r\mu_v - \gamma rw_5\right)^2$$

$$A_{22} = \gamma C_2kr - \gamma C_2k\mu + \gamma C_1C_2kNQ + C_1C_2kNQw_4 + \gamma rw_5 + C_1^2kNQw_5 + \gamma r\mu_v + C_1^2kNQ\mu_v - C_1C_2kNQ\mu_v$$

$A_{23} = -\frac{1}{2}\gamma kr + \frac{\gamma k\mu}{2} - \frac{1}{2}\gamma C_1kNQ + \frac{1}{2}C_1kNQw_4 + \frac{\gamma rw_5}{2C_2} + \frac{C_1^2kNQw_5}{2C_2} + \frac{1}{2}C_1kNQ\mu_v + \frac{\gamma r\mu_v}{2C_2} + \frac{C_1^2kNQ\mu_v}{2C_2}w_0 = (\alpha_1 + \pi)$, $w_1 = (\alpha_2 + \pi)$, $w_2 = (\alpha_1 + \pi + d)$, $w_3 = (\alpha_2 + \pi + d + \delta)$, $w_4 = \eta_1\frac{N_i}{N}$, $w_5 = \eta_2\frac{N_s}{N}$

Hence, $\lambda_1^* = \beta_1S^*\left(V_f^* + M^*\right)$ and $\lambda_2^* = \beta_1S^*\left(V_v^* + I^*\right)$ are forces of infestation and infection respectively.

It follows that from equation (14) $a_1 > 0$ (since all the model parameters are non-negative). Furthermore, $a_5 > 0$ whenever $R_0 < 1$. Thus, the number of positive real roots of the polynomial (14) can depends on the sign ofa_2, $a_3$anda_4. This can be analyzed using the Descartes rule of sign on quartic polynomial$f(x) = a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$, given in (14) with$(x = \lambda^*)$.

The various possibilities for the roots of $f(x)$are tabulated in Table 3.

The results of theorem 1 is deduced from the various possibilities highlighted in Table 3.

Theorem 1 of the model (1)-(6) claimed the following results:

1. has a unique endemic steady state if $R_0 > 1$and whenever cases 1, 2, 3 and 6 of Table 3 are satisfied
2. could have more than one endemic steady state if and whenever cases 4, 5, 7 and 8 of Table 3 are satisfied
3. could have 2 or more endemic steady states if $R_0 < 1$ and whenever cases 2-8 of Table 3 are satisfied

The existence of multiple endemic steady states of the model (1)-(6) when $R_0 < 1$ (as shown on Table 3) suggests the possibility of backward bifurcation to exist which is similar to [20].

Corollary 1 The model (1)-(6) undergoes backward bifurcation at $R_0 < 1$ from the possibilities shown in table 3

Theorem 2 The model (1)-(6) has a backward bifurcation *at* $R_0 < 1$ if and only if case ii and Case iii hold.

Proof

To show the existence of backward bifurcation in model (1)-(6) *at* $R_0 < 1$ if and only if case ii and Case iii hold, center manifold theorem is used as in [21]

Table 3. Number of possible positive real roots of $f(x)$ for $R_0 < 1$and $R_0 > 1$

Cases	a_1	a_2	a_3	a_4	a_5	R_0	Number of sign change	Number of possible real roots (endemic steady state)
1	+	+	+	+	+	$R_0 < 1$	0	0
	+	+	+	+	-	$R_0 > 1$	1	1
2	+	-	-	-	+	$R_0 < 1$	2	0,2
	+	-	-	-	-	$R_0 > 1$	1	1
3	+	+	-	-	+	$R_0 < 1$	2	0,2
	+	+	-	-	-	$R_0 > 1$	1	1
4	+	-	+	-	+	$R_0 < 1$	4	0,2,4
	+	-	+	-	-	$R_0 > 1$	3	1,3
5	+	-	-	+	+	$R_0 < 1$	2	0,2
	+	-	-	+	-	$R_0 > 1$	3	1,3
6	+	+	+	-	+	$R_0 < 1$	2	0,2
	+	+	+	-	-	$R_0 > 1$	1	1
7	+	+	-	+	+	$R_0 < 1$	2	0,2
	+	+	-	+	-	$R_0 > 1$	3	1,3
8	+	-	+	+	+	$R_0 < 1$	2	0,2
	+	-	+	+	-	$R_0 > 1$	3	1,3

Consider model (1)-(6)

$$\frac{dX}{dt} = G(X, \varphi) \tag{15}$$

Where φ is the bifurcation parameter and G is a continuously differentiable at least twice (both in x and φ).

Using the centre manifold theory, the following change of variables is made by denoting $S = x_1$, $M = x_2$, $I = x_3$, $V_f = x_4$, $V_v = x_5$ and $A_B = x_6$. The vector notation is given by$X = (x_1, x_2, x_3, x_4, x_5, x_6)^T$. Therefore, the model is written in a vector form as $\frac{dX}{dt} = G = (g_1, g_2, g_3, g_4, g_5, g_6)^T$ such that

$$\left.\begin{aligned}
\dot{x}_1(t) &= A - \beta_1 x_1 (x_4 + x_5 + x_2 + x_3) + (\alpha_1 + \pi) x_2 + (\alpha_2 + \pi) x_3 - d x_1 \\
\dot{x}_2(t) &= \beta_1 x_1 (x_4 + x_2) - \beta_2 x_2 (x_5 + x_3) - (\alpha_1 + \pi + d) x_2 \\
\dot{x}_3(t) &= \beta_1 x_1 (x_5 + x_3) + \beta_2 x_2 (x_5 + x_3) x_2 - (\alpha_2 + \pi + d + \delta) x_3 \\
\dot{x}_4(t) &= \gamma x_4 \left(1 - \tfrac{x_4}{Q(N)}\right) - (C_1 x_6 + \mu) x_4 \\
\dot{x}_5(t) &= \eta_1 x_4 \tfrac{N_i}{N} - \eta_2 x_5 \tfrac{N_s}{N} + \eta_3 - (C_2 x_6 + \mu) x_5 \\
\dot{x}_6(t) &= r x_6 \left(1 - \tfrac{x_6}{k}\right) + (C_1 x_4 + C_2 x_5) x_6 - \mu x_6
\end{aligned}\right\} \tag{16}$$

Let$\beta_1$denote a bifurcation parameter such that $\beta_1 = \beta_1^*$. The system is linearized at the disease-free and infestation-free steady stateE_1^*. Given the reproduction numbers

$R_{01} = \frac{A\beta_1}{d(\alpha_1+\pi+d)}$, $R_{02} = \frac{A\beta_1}{d(\alpha_2+\pi+d+\delta)}$and Max (R_{01}, R_{02}). Hence, the maximum between the two reproduction numbers, is given in equation (17).

$$R_{01} = \frac{A\beta_1}{d(\alpha_1 + \pi + d)} \tag{17}$$

Where $R_{01} = 1$is the bifurcation point from equation (17), we have:

$$\beta_1^* = \frac{d(\alpha_1 + \pi + d)}{A} \tag{18}$$

At the disease-free and infestation-free steady state, the Jacobian is given as

$$J(E_1^*) = \begin{bmatrix} -d & -b_1 & b_4 & -\beta_1\frac{A}{d} & -\beta_1\frac{A}{d} & 0 \\ 0 & b_2 & 0 & \beta_1\frac{A}{d} & 0 & 0 \\ 0 & 0 & b_3 & 0 & \beta_1\frac{A}{d} & 0 \\ 0 & 0 & 0 & b_5 & 0 & 0 \\ 0 & 0 & 0 & \eta_1\frac{N_i}{N} & b_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_7 \end{bmatrix} \tag{19}$$

Let's denote: $b_1 = \beta_1\frac{A}{d} + (\alpha_1 + \pi)$, $b_2 = \beta_1\frac{A}{d} - (\alpha_1 + \pi + d)$, $b_3 = \beta_1\frac{A}{d} - (\alpha_1 + \pi + d + \delta)$,$b_4 = \beta_1\frac{A}{d} + (\alpha_2 + \pi)$, $b_5 = \gamma - \mu$, $b_6 = -\left(\eta_2\frac{N_s}{N} + \mu\right)$ and $b_7 = \gamma - \mu$

The linearized system has a hyperbolic steady state (that is a linearized system has a simple zero eigenvalue and all other eigenvalues have negative real parts. Hence, the center manifold theory can be applied to the model. Next, we evaluate the left and right eigenvector which are associated with zero as a simple eigenvalue of equation (19). Suppose, $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)^T$are to be the right eigenvector associated with the eigenvalue zero, then the following equations are obtained

$$\left.\begin{array}{l} -d\omega_1 - b_1\omega_2 + b_4\omega_3 - \beta_1\frac{A}{d}\omega_4 - \beta_1\frac{A}{d}\omega_5 = 0 \\ b_2\omega_2 + \beta_1\frac{A}{d}\omega_4 = 0 \\ b_3\omega_3 + \beta_1\frac{A}{d}\omega_5 = 0 \\ b_5\omega_4 = 0 \\ \eta_1\frac{N_i}{N}\omega_4 + b_6\omega_5 = 0 \\ b_7\omega_6 = 0 \end{array}\right\} \tag{20}$$

Getting the solution of equations (20), we have:

$\omega_4 = \omega_6 = 0$, $\omega_1 = \beta_1\frac{A}{d}\omega_5$, $\omega_2 = -\frac{\beta_1 A}{b_1 d}\omega_5$, $\omega_3 = -\frac{\beta_1 A}{b_3 d}\omega_5$ and$\omega_5 = \omega_5 free$

Furthermore, the system of equation (19) with β_1^*as a bifurcation parameter has simple zero eigenvalues. Using the center manifold theory to analyzed the system nearβ_1^*, the Jacobian matrix nearβ_1^* has a left eigenvectors associated with the zero eigenvalues given by $v = (v_1, v_2, v_3, v_4, v_5, v_6)$for the dynamical system (19). Solving the left eigenvector given by $v = (v_1, v_2, v_3, v_4, v_5, v_6)$and satisfying$w.v = 1$, we take the transpose of the system (20) to obtain below:

$$[JE_1^*]^T = \begin{bmatrix} -d & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 & 0 \\ b_4 & 0 & b_3 & 0 & 0 & 0 \\ -\beta_1\frac{A}{d} & \beta_1\frac{A}{d} & 0 & b_5 & \eta_1\frac{N_i}{N} & 0 \\ -\beta_1\frac{A}{d} & 0 & \beta_1\frac{A}{d} & 0 & b_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

Equation (21) is translated into equations (22)

$$\left.\begin{array}{l} -dv_1 = 0 \\ b_1v_1 + b_2v_2 = 0 \\ b_4v_1 + b_3v_2 = 0 \\ -\beta_1\frac{A}{d}v_1 + \beta_1\frac{A}{d}v_2 + b_5v_4 + \eta_1\frac{N_i}{N}v_5 = 0 \\ \beta_1\frac{A}{d}v_1 + \beta_1\frac{A}{d}v_3 + b_6v_5 = 0 \\ b_7v_6 = 0 \end{array}\right\} \tag{22}$$

Solving the system of equation (22), we have:

$$v_1 = v_2 = v_6 = 0$$

$v_3 = -\frac{db_6}{A\beta_1}v_5$, $v_4 = \frac{\eta_1 N_i}{Nb_5}v_5$ and $v_5 = v_5 free$

Computation of bifurcation Coefficient's sign a and b

When we apply the center manifold theorem by using Theorem 4.1 in [21] the value of a and $b$are computed for the direction of bifurcation as follows:

Since$v_1 = v_2 = v_6 = 0$ for$k = 1, 2, 6$, then, the values of $k = 3, 4, 5$is considered and employed in the computation process. From the system of equation (16), we observe that the second partial derivatives of $f_4$and $f_5$are zeros. Hence, the only contributing terms are those corresponding tof_3. Thus, the associated nonzero second order partial derivatives at disease and infestation free steady state are given as:

$\frac{\partial^2 f_3}{\partial x_3 \partial x_1} = \beta_1$, $\frac{\partial^2 f_3}{\partial x_3 \partial x_2} = \beta_2$, $\frac{\partial f_3}{\partial x_5 \partial x_1} = \beta_1$ and $\frac{\partial f_3}{\partial x_5 \partial x_2} = \beta_2$

$$a = v_3 \sum_{i=j=k}^{5} \omega_i \omega_j \frac{\partial^2 f_3}{\partial x_i \partial x_j}(0,0) + v_4 \sum_{i=j=k}^{5} \omega_i \omega_j \frac{\partial^2 f_4}{\partial x_i \partial x_j}(0,0) + v_5 \sum_{i=j=k}^{5} \omega_i \omega_j \frac{\partial^2 f_5}{\partial x_i \partial x_j}(0,0)\, a$$

$$= v_3 \left(\omega_3 \omega_1 \frac{\partial^2 f_3}{\partial x_3 \partial x_1} + \omega_3 \omega_2 \frac{\partial^2 f_3}{\partial x_3 \partial x_2} + \omega_5 \omega_1 \frac{\partial^2 f_3}{\partial x_5 \partial x_1} + \omega_5 \omega_2 \frac{\partial^2 f_3}{\partial x_5 \partial x_2} \right) + v_4 (0) + v_5 (0)$$

$$a = \frac{-db_6}{A\beta_1} \left[\left(-\beta_1 \frac{A}{d} \omega_5 \right) \left(\beta_1 \frac{A}{d} \omega_5 \right) \beta_1 + \left(-\beta_1 \frac{A}{d} \omega_5 \right) \left(-\beta_1 \frac{A}{d} \omega_5 \right) \beta_2 + \left(\omega_5 \beta_1 \frac{A}{d} \omega_5 \beta_1 \right) + \omega_5 \beta_1 \left(-\beta_1 \frac{A}{d} \omega_5 \right) \right]$$

$$a = \frac{-db_6}{A\beta_1} \left[-\omega_5^2 \left(\beta_1 \frac{A}{d} \right)^2 \beta_1 + \omega_5^2 \left(\beta_1 \frac{A}{d} \right)^2 \beta_2 + \omega_5^2 \beta_1^2 \frac{A}{d} - \omega_5^2 \beta_1 \frac{A}{d} \beta_2 \right]$$

$$a = -b_6 \omega_5^2 \left[-\beta_1 \frac{A}{d} (\beta_1 - \beta_2) + 1 (\beta_1 - \beta_2) \right]$$

$a = b_6 \omega_5^2 \left[\left(\beta_1 \frac{A}{d} + 1 \right) (\beta_1 - \beta_2) \right] > 0$, if $\beta_1 > \beta_2$

$$b = v_3 \sum_{i=j=2}^{5} \omega_5 \frac{\partial^2 f_3}{\partial x_3 \partial \beta_1}(0,0) + v_4 \sum_{i=j=2}^{5} \omega_5 \frac{\partial^2 f_3}{\partial x_4 \partial \beta_1}(0,0) + v_5 \sum_{i=j=2}^{5} \omega_5 \frac{\partial^2 f_3}{\partial x_5 \partial \beta_1}(0,0)$$

$$b = v_3 \omega_5 + v_5 \omega_5$$

$$b = \omega_5 (v_3 + v_5) = \omega_5 \left(-\frac{db_6}{A\beta_1} v_5 + v_5 \right) = \omega_5 v_5 \left(1 - \frac{db_6}{A\beta_1} \right)$$

$b = \omega_5 v_5 \left(1 - \frac{db_6}{A\beta_1} \right) > 0$, where $1 > \frac{db_6}{A\beta_1}$ and v_5 is a free variable, strictly positive.

Since $a > 0$and $b > 0$, the system of equation (1)-(6) exhibit a backward bifurcation. Recall [22] and [23] critically decomposed the center manifold and found that the sign of "a" can be used to determine the direction of the bifurcation. Hence, by this reason, we conclude that the system of equation (1)-(6) exhibits backward bifurcation.

Theorem 3: The model undergoes a backward bifurcation since$a > 0$, $b > 0$which occurs at $R_0 = 1$, if $\beta_1 < 0$, this implies that there exists unstable negative endemic steady state and when$\beta_1 > 0$, it implies that there exists stable positive endemic steady state. Therefore, the endemic steady state or (EEP) is locally asymptotically stable for R_0 close to one.

3.2. Global stability of the endemic steady state of the model

The global asymptotic stability of endemic steady state E_4^* of the model (1)-(6) is presented in a special case where honeybee receiving treatment does not transmit disease. Also, it is further assumed that the efficacy of treatment is perfect at$\pi = 1$. When this holds, then, backward bifurcation ceased to exist. The model (1)-(6) with special case $\pi = 1$is given in equation (23) below:

$$\left.\begin{aligned}
\dot{S}(t) &= A - \beta_1 S \left(V_f + V_v + M + I \right) + (\alpha_1 + 1) M + (\alpha_2 + 1) I - dS \\
\dot{M}(t) &= \beta_1 S \left(V_f + M \right) - \beta_2 M (V_v + I) - (\alpha_1 + 1 + d) M \\
\dot{I}(t) &= \beta_1 S (V_v + I) + \beta_2 M (V_v + I) - (\alpha_2 + 1 + d + \delta) I \\
\dot{V}_f(t) &= \gamma V_f \left(1 - \frac{V_f}{Q(N)} \right) - (C_1 A_B + \mu_v) V_f \\
\dot{V}_v(t) &= \eta_1 V_f \frac{N_i}{N} - \eta_2 V_v \frac{N_v}{N} + \eta_3 - (C_2 A_B + \mu_v) V_v \\
\dot{A}_B(t) &= \tau A_B \left(1 - \frac{A_B}{k} \right) + \left(C_1 V_f + C_2 V_v \right) A_B - \mu A_B
\end{aligned}\right\} \tag{23}$$

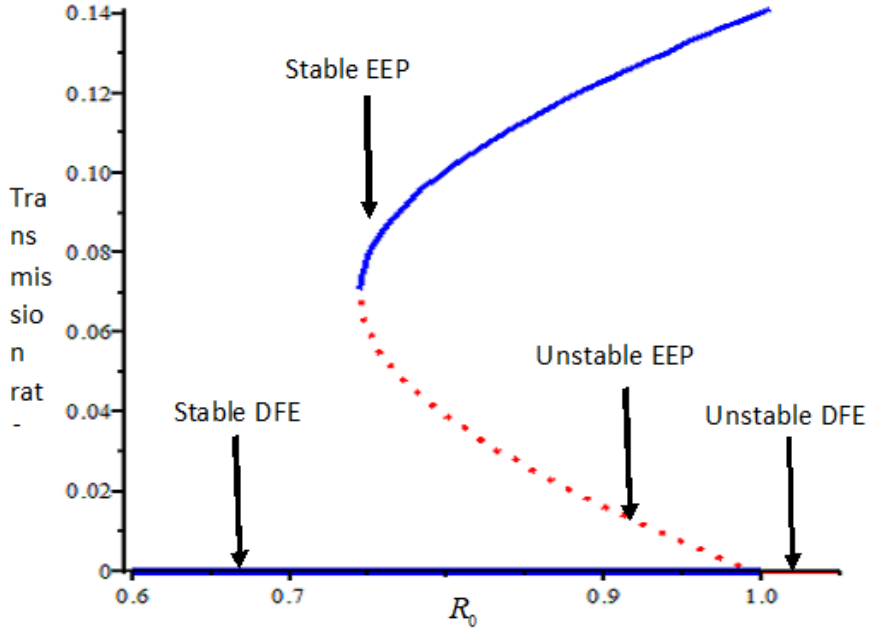

Figure 1. Backward bifurcation at $R_0 < 1$of the model

In this section, we need to establish the global stability of the endemic steady stateE_4^* of the reduced dynamical system (23). We now construct positive lyapunov function $V(t)$by Goh-Volterra type.

Theorem 4*:* The endemic steady state of reduced model (23), which exists if $R_0 > 1$is Globally Asymptotically Stable (GAS)

Proof: Let $R_0 > 1$and consider a non-linear lyapunov fuction of Goh-Volterra type given by

$$\begin{aligned} V = &\left(S - S^* - S^* \log \tfrac{S}{S^*}\right) + \left(M - M^* - M^* \log \tfrac{M}{M^*}\right) + \left(I - I^* - I^* \log \tfrac{I}{I^*}\right) + \left(V_f - V_f{}^* - V_f{}^* \log \tfrac{V_f}{V_f{}^*}\right) \\ &+ \left(V_v - V_v{}^* - V_v{}^* \log \tfrac{V_v}{V_v{}^*}\right) + \left(A_B - A_B{}^* - A_B{}^* \log \tfrac{A_B}{A_B{}^*}\right) \end{aligned} \tag{24}$$

Differentiating with respect to time, we have:

$$\dot{V} = \left(\dot{S} - \frac{S^*\dot{S}}{S}\right) + \left(\dot{M} - \frac{M^*\dot{M}}{M}\right) + \left(\dot{I} - \frac{I^*\dot{I}}{I}\right) + \left(\dot{V}_f - \frac{\dot{V}_f V_f{}^*}{V_f}\right) + \left(\dot{V}_v - \frac{V_v{}^*\dot{V}_v}{V_v}\right) + \left(\dot{A}_B - \frac{A_B{}^*\dot{A}_B}{A_B}\right) \tag{25}$$

Substituting the derivatives of $\left(S, M, I, V_f, V_v, A_B\right)$from the model (1)-(6) into (25), we have:

$$\begin{aligned} \dot{V} = &\left(A - \beta_1 S\left(V_f + V_v + M + I\right) + w_0 M + w_1 I - dS - \tfrac{S^* A}{S} + \beta_1 S^*\left(V_f + V_v + M + I\right) + w_0 M + w_1 I + dS^*\right) \\ &+ \left(\beta_1 S\left(V_f + M\right) - \beta_2 M\left(V_v + I\right) - w_2 M - \tfrac{M^*}{M}\beta_1 S\left(V_f + M^*\right) + \beta_2 M^*\left(V_v + I\right) + w_2 M^*\right) \\ &+ \left(\beta_1 S\left(V_v + I\right) + \beta_2 M\left(V_v + I\right) - w_3 I - \tfrac{I^*}{I}\beta_1 S\left(V_v + I^*\right) + \beta_2 M\left(V_v + I^*\right) + w_3 I^*\right) \\ &+ \left(\gamma V_f\left(1 - \tfrac{V_f}{Q(N)}\right) - C_1 A_B V_f + \mu_v V_f - \tfrac{V_f{}^*}{V_f}\gamma V_f{}^*\left(1 - \tfrac{V_f{}^*}{Q(N)}\right) + C_1 A_B V_f{}^* + \mu_v V_f{}^*\right) \\ &+ \left(\eta_1 V_f \tfrac{N_i}{N} - \eta_2 V_v \tfrac{N_s}{N} + \eta_3 - C_2 A_B V_v - \mu_v V_v - \tfrac{V_v{}^*}{V_v}\eta_1 V_f \tfrac{N_i}{N} + \eta_2 V_v{}^* \tfrac{N_s}{N} + \eta_3 + C_2 A_B V_v{}^* + \mu_v V_v{}^*\right) \\ &+ \left(rA_B\left(1 - \tfrac{A_B}{k}\right) + \left(C_1 V_f + C_2 V_v\right) A_B - \mu A_B - \tfrac{A_B{}^*}{A_B} rA_B{}^*\left(1 - \tfrac{A_B{}^*}{k}\right) + \left(C_1 V_f + C_2 V_v\right) A_B{}^* + \mu A_B{}^*\right) \end{aligned} \tag{26}$$

Where$w_0 = (\alpha_1 + 1)$, $w_1 = (\alpha_2 + 1)$, $w_2 = (\alpha_1 + 1 + d)$and $w_3 = (\alpha_1 + 1 + d + \delta)$.

Using the relation below:

$$A = \beta_1 S^* \left(V_f^* + V_v^* + M^* + I^*\right) + dS^* - w_0 M^* - w_1 I^* \tag{27}$$

Hence, equation (27) is at steady state from equations (1)-(6). Therefore, substituting equation (27) into equation (26), we have:

$$\begin{aligned}
\dot{V} = &\left(\begin{array}{l} \beta_1 S^* \left(V_f^* + V_v^* + M^* + I^*\right) + dS^* - w_0 M^* - w_1 I^* - \beta_1 S \left(V_f + V_v + M + I\right) + w_0 M + w_1 I - dS \\ -\frac{S^*}{S}\beta_1 S^* \left(V_f^* + V_v^* + M^* + I^*\right) - \frac{S^*}{S} dS^* + w_0 \frac{S^*}{S} M^* w_1 \frac{S^*}{S} I^* + \beta_1 S^* \left(V_f + V_v + M + I\right) + w_0 M + w_1 I + dS^* \end{array} \right) \\
&+ \left(\beta_1 S \left(V_f + M\right) - \beta_2 M \left(V_v + I\right) - w_2 M - \frac{M^*}{M}\beta_1 S \left(V_f + M^*\right) + \beta_2 M^* \left(V_v + I\right) + w_2 M^*\right) \\
&+ \left(\beta_1 S \left(V_v + I\right) + \beta_2 M \left(V_v + I\right) - w_3 I - \frac{I^*}{I}\beta_1 S \left(V_v + I^*\right) + \beta_2 M \left(V_v + I^*\right) + w_3 I^*\right) \\
&+ \left(\gamma V_f \left(1 - \frac{V_f}{Q(N)}\right) - C_1 A_B V_f - \mu_v V_f - \frac{V_f^*}{V_f}\gamma V_f^* \left(1 - \frac{V_f^*}{Q(N)}\right) + C_1 A_B V_f^* + \mu_v V_f^*\right) \\
&+ \left(\eta_1 V_f \frac{N_i}{N} - \eta_2 V_v \frac{N_s}{N} + \eta_3 - C_2 A_B V_v - \mu_v V_v - \frac{V_v^*}{V_v}\eta_1 V_f \frac{N_i}{N} + \eta_2 V_v^* \frac{N_s}{N} + \eta_3 + C_2 A_B V_v^* + \mu_v V_v^*\right) \\
&+ \left(rA_B \left(1 - \frac{A_B}{k}\right) + \left(C_1 V_f + C_2 V_v\right) A_B - \mu A_B - \frac{A_B^*}{A_B} rA_B^* \left(1 - \frac{A_B^*}{k}\right) + \left(C_1 V_f + C_2 V_v\right) A_B^* + \mu A_B^*\right)
\end{aligned} \tag{28}$$

Further simplification and collecting all the infested and infected compartments without star (*) from equation (28) and equating to zero, we have

$$\beta_1 S^* \left(V_f + V_v + M + I\right) - \beta_2 M \left(V_v + I\right) - w_3 I - \mu_v V_f - \mu_v V_v \tag{29}$$

A little perturbation of steady state from equation (1)-(6) and (29), we have:

$$w_3 = \beta_1 S^* - \beta_2 M \text{and} \mu_v = \beta_1 S^* \tag{30}$$

Substituting equation (30) into equation (29) with some algebraic calculations give:

$$\begin{aligned}
\dot{V} = &\left(\begin{array}{l} \beta_1 S^* \left(V_f^* + V_v^* + M^* + I^*\right) + dS^* - w_0 M^* - w_1 I^* - dS - \frac{S^{*2}}{S}\beta_1 \left(V_f^* + V_v^* + M^* + I^*\right) - \frac{S^{*2}}{S} d + w_0 \frac{S^*}{S} M^* \\ +w_1 \frac{S^*}{S} I^* + 2w_0 M + 2w_1 I + dS^* \end{array} \right) \\
&+ \left(-w_2 M - \frac{M^*}{M}\beta_1 S \left(V_f + M^*\right) + \beta_2 M^* \left(V_v + I\right) + w_2 M^*\right) + \left(-w_3 I - \frac{I^*}{I}\beta_1 S \left(V_v + I^*\right) + \beta_2 M \left(V_v + I^*\right) + w_3 I^*\right) \\
&+ \left(\gamma V_f \left(1 - \frac{V_f}{Q(N)}\right) - \frac{V_f^{*2}}{V_f}\gamma \left(1 - \frac{V_f^*}{Q(N)}\right) + C_1 A_B V_f^* + \mu_v V_f^*\right) \\
&+ \left(-\eta_2 V_v \frac{N_s}{N} + \eta_3 + \eta_2 V_v^* \frac{N_s}{N} + \eta_3 + C_2 A_B V_v^*\right) \\
&+ \left(rA_B \left(1 - \frac{A_B}{k}\right) + -\mu A_B - \frac{A_B^{*2}}{A_B} r \left(1 - \frac{A_B^*}{k}\right) + \left(C_1 V_f + C_2 V_v\right) A_B^* + \mu A_B^*\right)
\end{aligned} \tag{31}$$

Factorizing equation (31), we have:

$$\dot{V} = dS^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \beta_1 S^* \left(V_f^* + V_v^* + M^* + I^*\right) \left(1 - \frac{S^*}{S}\right) + w_1 S^* I^* \left(\frac{1}{S} - \frac{\beta_1}{w_1} - \frac{w_2}{S^* w_0}\right) + w_0 M^* S^* \left(\frac{1}{S} - \frac{1}{S^*} - \frac{w_2}{S^* w_0}\right) \tag{32}$$

Thus, since the arithmetic mean exceeds the geometric mean, the following inequalities from equation (32) hold

$\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) \leq 0, \left(1 - \frac{S^*}{S}\right) \leq 0, \left(\frac{1}{S} - \frac{\beta_1}{w_1} - \frac{w_2}{S^* w_0}\right)$and $\left(\frac{1}{S} - \frac{1}{S^*} - \frac{w_2}{S^* w_0}\right) \leq 0$

Therefore, $\dot{V} \leq 0$ for $R_0 > 1$. Hence, $V$is a lyapunov function in $\Psi$and it follows by Lassalle's invariance principle, that every solution to the modified model (1)-(6) approaches the associated endemic steady state of reduced model (23) as $t \to \infty$ for $R_0 > 1$. Thus, the endemic steady state is Globally Asymptotically Stable (GAS)

3.3. Sensitivity analysis of the model

Sensitivity analysis is a scientific method employed to determine how different values of an independent variable will affect a particular dependent variable under a certain set of hypotheses. It can also be used to determine which parameters are most significant in the model output [24]. We hereby adopted [25] technique in carrying out the sensitivity analysis, by obtaining the normalized sensitivity index of c with respect to individual parameters. Therefore, we consider the parameters that appeared in reproduction numbers in equation (17) which invariably affect equation (1)-(6) if changes in their values occur.

Table 4. Signs of Sensitivity Indices for two reproduction numbers R_{01} and R_{02}

Parameters	Elasticity Index	Sensitivity Index
A	1	positive
β_1	1	positive
d	0.456116279	Positive
α_1	-0.279069767	Negative
π	-0.627906976	Negative
α_2	-0.163265306	Negative
δ	-0.122448979	Negative

Consequently, sensitivity indices of the basic reproduction numbers in equation (17) with respect to the model parameter are obtained. The computations are as follows:

$$\text{Sensitivity index for } A: \Gamma_A^{R_0} = \frac{\partial R_0}{\partial A} \times \frac{A}{R_0} = \frac{\beta_1}{d\left(\alpha_1+\pi+d\right)} \times \frac{A}{1} \times \frac{d\left(\alpha_1+\pi+d\right)}{A\beta_1} = 1, \Gamma_A^{R_{01}} = 1 \tag{33}$$

$$\text{Sensitivity index for } \beta_1: \Gamma_A^{R_0} = \frac{\partial R_0}{\partial \beta_1} \times \frac{\beta_1}{R_0} = \frac{A}{d\left(\alpha_1+\pi+d\right)} \times \frac{\beta_1}{1} \times \frac{d\left(\alpha_1+\pi+d\right)}{A\beta_1} = 1, \Gamma_{\beta_1}^{R_{01}} = 1 \tag{34}$$

$$\text{Sensitivity index for } d: \Gamma_d^{R_0} = \frac{\partial R_0}{\partial \beta_1} \times \frac{\beta_1}{R_0} = \frac{A\beta_1}{d^2\left(\alpha_1+\pi+d\right)^2} \times \frac{d}{1} \times \frac{d\left(\alpha_1+\pi+d\right)}{A\beta_1} = \frac{1}{\left(\alpha_1+\pi+d\right)}, \Gamma_d^{R_0} = \frac{1}{\left(\alpha_1+\pi+d\right)} \tag{35}$$

$$\text{Sensitivity index for } \alpha_1: \Gamma_{\alpha_1}^{R_0} = \frac{\partial R_0}{\partial \beta_1} \times \frac{\alpha_1}{R_0} = \frac{A\beta_1}{d\left(\alpha_1+\pi+d\right)^2} \times \left(-\frac{\alpha_1}{1}\right) \times \frac{d\left(\alpha_1+\pi+d\right)}{A\beta_1} = -\frac{\alpha_1}{\left(\alpha_1+\pi+d\right)}, \Gamma_{\alpha_1}^{R_0} = -\frac{\alpha_1}{\left(\alpha_1+\pi+d\right)} \tag{36}$$

$$\text{Sensitivity index for } \pi: \Gamma_\pi^{R_0} = \frac{\partial R_0}{\partial \beta_1} \times \frac{\pi}{R_0} = \frac{A\beta_1}{d\left(\alpha_1+\pi+d\right)^2} \times \left(-\frac{\pi}{1}\right) \times \frac{d\left(\alpha_1+\pi+d\right)}{A\beta_1} = -\frac{\pi}{\left(\alpha_1+\pi+d\right)}, \Gamma_\pi^{R_0} = -\frac{\pi}{\left(\alpha_1+\pi+d\right)} \tag{37}$$

The signs of sensitivity indices explicitly indicate whether reproduction number increases (positive sign) or decrease (negative sign) with the model parameters. Then, establishing the various sensitivity indices we have equations (33)-(37) which are presented in Table 4.

Thus, from the Table 4, we clearly examined that the sensitivity analysis of both R_{01} and R_{02} with respect to the model parameters computed are either of positive or negative sign. The model parameters with positive sign are: constant recruitment rate of healthy honeybee, rate of infestation by virus-free *Varroa-mite* and natural death rate for all honeybee population. Conversely, the model parameters with negative sign are: disinfestation rate for honeybee infested by virus-free *Varroa-mite*, treatment, disinfestation rate for honeybee infested by virus-carrying *Varroa-mite* and infection induced death for the honeybee population. The model parameter with positive sign indicated that, any attempt to increase the value of these parameters will increase the spread of *Varroosis*, as such; they have a greater impact on the rise of the reproduction number. On the contrary, increasing the value of model parameter with negative sign indicated that, there is invariable decrease in the spread of *Varroosis*, which sufficiently reduces the reproduction number.

4. Conclusion

A mathematical model, which considers treatment and biocontrol as combine intervention strategies adopted to control the menace of *Varroosis* in honeybee colony, is developed and analyzed. The study established the existence of disease-free and infestation-free, infestation with virus-free *Varroa-mite*, infestation with virus-carrying *Varroa-mite* and endemic steady states respectively. The centre manifold theory was used to determine the local asymptotic stability of the endemic steady state. It revealed that, model (1)-(6) undergoes backward bifurcation when its corresponding reproduction number$R_0 < 1$. This occurs due to ineffective treatment (i.e. $\pi \neq 1$). However, when treatment is 100%

effective (i.e. $\pi = 1$), the disease-free and infection-free steady state of model (1)-(6) is globally asymptotically stable provided that its associated reproduction number is less than unity. Also, the endemic steady state for a special case (i.e. the reduced model given by (23)) is shown to be globally asymptotically stable whenever its associated reproduction number is greater than unity ($R_0 > 1$).

As part of possible extension, more refined models with seasonality terms can be studied. Stochastic model mimics and portrays the phenomenal complexity nature of *Varroosis* rather than deterministic model. Also, Fractional derivatives, instead of integer-order derivatives to consider the long-term memory effect, with functional response II in honeybee is a possible extension of this study.

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