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Factorization in Phase-Space Finite Geometry and Weak Mutually Unbiased Bases

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Abstract

A phase-space factorization of lines in finite geometry $G(m)$ with variables in Z_m and its correspondence in finite Hilbert space $H(m)$ for m a non-prime was discussed. Using the method of Good [15], lines in $G(m)$ were factorized as products of lines $G(m_i)$ where m_i is a prime divisor of m. A lattice was formed between the non trivial sublines of $G(m)$ and lines of $G(m_i)$ and between a subspace of $H(m)$ and bases of $H(m_i)$ and existence of a link between lines in phase space finite geometry and bases in Hilbert space of finite quantum systems was discussed.

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1. Introduction

Finite geometry has received a lot of attention in the past. In particular, near-linear types are attracting attention. The reason may be related to its wide range of recently discovered uses. Lines of finite geometry are linked to phase-space finite quantum systems in the sense that for instance, taking the absolute value of the scalar product of any two (orthogonal) vectors each from different bases yields $\frac{1}{\sqrt{m}}$. Also, the number of mutually unbiased bases in a finite-dimensional Hilbert space is equal to $m + 1$. This number corresponds to the number of finite-dimensional

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lines in finite geometry [1–9]. In more recent times, non-near-linear finite geometry started receiving audience from researchers [11–14]. This could be linked to its duality with the weak mutually unbiased bases in finite quantum systems with variables in Z_m.

For a prime dimensional finite geometry, two lines intersect at a point. For a non-prime dimensional finite geometry, two lines intersect at least one point. The Geometry of this type is called the non-near-linear finite geometry [12–14]. In this article, we show how to decompose a large-dimensional finite geometry, called nonlinear finite geometry, into a product of many prime finite dimensional geometries, called near-linear geometries. This approach was adapted from Good's method of Fast Fourier Transforms [15]. This method decomposes the large Hilbert space qudits into smaller qudits using an appropriate uniform transformation. This method arose because of the difficulty of solving problems consisting of very large integers. In [15] a large integer was factored as the product of many small integers. The same was employed in [10] to factor a large-dimensional finite quantum system with variables of Z_m as a product of many small-dimensional finite quantum systems. For $m_i \mid m$; $G_{m_i} \subset G_m$, lines in G_m is factored as as products of lines in $G_{m_i}s'$, each lines in $G_{m_i}s'$ has m_i points. For $m_k, m_i \mid m$; the union of any two lines in G_{m_i} produce a line in G_{m_k} with m_k points. Hence, form a lattice between lines in phase-space finite geometry G_m, for $G_{m_i} \mid G_m$. We divide the whole work into the following parts; Various notations used throughout the work were defined in the preliminaries of this work in section II. Section III covers finite geometry $G(m)$. We discuss the factorization of lines in finite geometry in section IV. Section V gave a symplectic structure on $G(m)$ with numerical examples. In section VI of this work, the non-prime dimensional finite quantum systems $\mho(m)$ with variables in Z_m where m is a non-prime is expressed as products in prime dimensional finite quantum systems $\mho(m_i)$ with variables in Z_{m_i} where p is a prime. In section VII, we relate the concept of factorization to mutually unbiased bases in prime dimensional Hilbert space. Section VIII covers the extension of mutually unbiased bases where a non-prime dimensional Hilbert space is expressed as a product of two prime dimensional Hilbert space $H(m)$ via factorization. The final section draws a necessary conclusion from the study.

2. Preliminaries

(i) Let Z_m represents the ring of integer modulo m.

(ii) $|Z_m^*|$ represents the invertible integer modulo m. $|Z_m|$ is $\varphi(m)$. Where

$$\varphi(m) = m \prod_{j=1}^{k} \left(1 - \frac{1}{m_j}\right) \tag{1}$$

(iii) The Dedekind psi function $\psi(m)$ is defined as

$$\psi(m) = m \prod_{j=1}^{k} \left(1 + \frac{1}{m_j}\right); m_j = \text{prime} \tag{2}$$

(iv) The set of divisor is denoted in this work by $\{D(m)\}$. Its cardinality is a divisor function $\sigma_0(m)$. Here $m_i|m$ means m_i divides m. If $m_i|m$ it means there exists a number say k an integer such that $\frac{m}{m_i} = k$ that is $m = km_i$. We showed the existence of a bijection between the products of the distinct set $\{D(m)\}$ of prime divisors m and Z_{m_i}. The elements of Z_{m_i} are embedded in Z_m for $m_i|m$ thus

$$Z_{m_i} \ni \zeta \to Z_m \ni \frac{m\zeta}{m_i}. \tag{3}$$

(v) $GCD(\xi, \rho)$ represents the greatest common divisor of two elements ξ and ρ.

(vi) Integer m is expressed as products of its distinct primes

$$m = m_1 \times m_2 \times ... \times m_k. \tag{4}$$

In this work, our discussion centres on a composite integer which express as products of two prime integers that is $m = m_1 \times m_2$. Z_m is a cyclic module.

3. Lines in finite geometry $G(m)$

A finite geometry $G(m) = Z_m \times Z_m$ is defined as the combination

$$G(m) = (P(m), L(m)). \tag{5}$$

P_m represent points on a line and $L(m)$ represent lines in $G(m)$ where

$$P(m) = \{(k, g)|k, g \in Z_m\}. \tag{6}$$

Definition 3.1. *A line $L(x, y)$ of $G(m)$ defined as*

$$L(x, y) = \{(\alpha x, \alpha y)|\ x, y \in Z_m)\}\ \alpha \in Z_m \tag{7}$$

The representation $\prod_{j=1}^{k} G(m_j)$ and $\prod_{j=1}^{k} Z_m \times Z_m$ have similar interpretation, so at times we interchange them.

We discuss extensively finite geometry as a result our point of focus is on both near-linear and non-near-linear geometry. Here, two lines intersect in at least one point.

Proposition 3.2. *(i) In $G(m)$ there exists $\psi(m)$ maximal lines with exactly m points.*

(ii) For $\eta \in Z_m^*$

$$L(\eta l, \eta v) = L(l, v). \tag{8}$$

also, if

$$For\ Z_m^{*\prime} \ni \eta\ then\ L(\eta l, \eta v) mod(m) \subset L(l, v). \tag{9}$$

(iv) if $GCD(l, v) \in Z_m^*), L(l, v)$ is a maximal line in $G(m)$. and if $GCD(l, v) \in Z_m - Z_m^*$ then $L(l, v)$ is a subline in $G(m)$

(iii) There exists $\psi(m)$ maximal lines in $G(m)$.

(iv) Suppose $G(m)$ is a finite geometry in equation(7). Then the line

$$L(l, v) = L(kl, kv) = \{(k\eta l, k\eta v)|k \in Z_m\},\quad in\ G(kv) \tag{10}$$

A line $L(kl, kv)$ in $G(kv)$ is a subline of

$$L(l, v) = \{(k'l, k'v)|\ k' = 0, ..., \eta v - 1\}, \tag{11}$$

(v) For $m_i|t$ two maximal lines have m_i points in common. The t points gives a subline $L(l, v)$ where $l, v \in \frac{m}{k} Z_{m_i}$.

4. Factorization of lines in finite geometry

Lines in $Z_m \times Z_m$ were factored as products of lines in $\prod_{i=1}^{k} Z_{m_i} \times Z_{m_i}$ then a one-to-one and onto map was established between lines in $G(m)$ and its factor lines in $G(m)$. We adopted this concept from Good [15]. Similar thought was applied in [1] and [11–13] to factorize large finite-dimensional quantum systems as products of its finite subsystems. Here we used the same approach to create the ordinates of each of the points on the lines $G(m)$ in non-near-linear geometries as products of many ordinates in the lines $G(m)$ in near-linear geometries. This was carried out by creating two bijections, one for each of the two ls' and ms' ordinates for each line thus:

$$l \longleftrightarrow (l_1, ..., l_k); l_j = l(mod\ m_j); l = \sum l_j s_j \tag{12}$$

$$m \longleftrightarrow (\bar{m}_1, ..., \bar{m}_k); \bar{m}_j = mt_{\mathbf{j}} = m_j t_j (mod\ m_j); m = \sum \bar{m}_j r_j (mod\ m) \tag{13}$$

where

$$r_j = \frac{m}{m_j}; t_j r_j = 1(mod\ m_j); s_j = t_j r_j \in Z_m. \tag{14}$$

l and m ordinates in the non-near-linear geometry we factorised in line with equations (12) and (13). Hence an existence of $1 - 1$ correspondence was confirmed between

$$L(l, m) \text{ in } G(m) \tag{15}$$

and lines

$$L(l_1, \bar{m}_1) \times ... \times L(l_k, \bar{m}_k) \in \prod_{i=1}^{k} (Z_{m_i} \times Z_{m_i}) \tag{16}$$

where

$$(l, m) \leftrightarrow (l_1, \bar{m}_1) \times ... \times (l_k, \bar{m}_k) \text{ and } m_j \text{ a prime} \tag{17}$$

In the previous work of [11–13] we confirm the following:

(i) $m\psi(m)$ maximal lines in total.

(ii) $\psi(m)$ distinct maximal lines.
In addition,

(iii) We found an existence of $\psi(\frac{m}{m_i})$ sublines each with m_i points.

Analogously, we observe the following

(i) $L(a, \bar{b})$ are prime factor lines of $Z_{m_i} \times Z_{m_i}$, where m_i is a prime number.

(ii) Lines in $Z_m \times Z_m = \prod_{j=1}^{k} (Z_{m_j} \times Z_{m_j})$ is related to expressing a non prime integer as products of its prime.

(iii) The subline $G(m_i)$ is related to the divisor m_i of an integer.

As an illustration, we express all maximal lines in $G(m) = Z_m \times Z_m$ for $m = 14$ in terms of its primes discussed in equations (12) and (13) above by decomposing line $L(2, 5)$.
Using equation (12) the ordinate 2 in $L(2, 5)$ is decomposed as;

$$2 \longleftrightarrow (0, 2) \tag{18}$$

also using equation (13) the ordinate 5 in $L(2, 5)$ is decomposed as;

$$5 \longleftrightarrow (1, 6) \tag{19}$$

Therefore $L(2, 5)$ is decomposed as;

$$L(0, 1) \times L(2, 6). \tag{20}$$

If we relate equation (20) to equations (12) and (13), $L(2, 5)$ is expressed as

$$L(1, -1) \times L(1, 2) \equiv \Omega(-1, 3). \tag{21}$$

Table 1. Factorization of Lines in G_6 as products of lines in G_2 and G_3 that is $G_6 = G_2 \times G_3$.

G_6	G_2	G_3
$L_6(0,1)$	$L_2(0,1)$	$L_3(0,2)$
$L_6(1,0)$	$L_2(1,0)$	$L_3(1,0)$
$L_6(1,1)$	$L_2(1,1)$	$L_3(1,2)$
$L_6(1,2)$	$L_2(1,0)$	$L_3(1,1)$
$L_6(1,3)$	$L_2(1,1)$	$L_3(1,0)$
$L_6(1,4)$	$L_2(1,0)$	$L_3(1,2)$
$L_6(1,5)$	$L_2(1,1)$	$L_3(1,1)$
$L_6(2,1)$	$L_2(0,1)$	$L_3(2,2)$
$L_6(2,3)$	$L_2(0,1)$	$L_3(2,0)$
$L_6(2,5)$	$L_2(0,1)$	$L_3(2,1)$
$L_6(3,1)$	$L_2(1,1)$	$L_3(0,2)$
$L_6(3,2)$	$L_2(1,0)$	$L_3(0,1)$

5. $Sp(2, Z_m)$ Transformation on $G(m)$

The matrix $M(f, g|n, l)$

$$M(f, g|n, l) \equiv \begin{pmatrix} f & g \\ n & l \end{pmatrix} where\ f, g, n, l \in Z_m\ and\ |M| = 1(mod m) \tag{22}$$

form a Symplectic group.

$$M(f, g|n, l)(\mu_i, \nu_i)^T = L(f\mu_i + g\nu_i, \mu_i n + l\nu_i), i = 1, 2, \cdots, m \tag{23}$$

As an illustration, acting a matrix $M(0, 1| - 1, l)$, on a line $L(1, l)$ this produces $\psi(m)$ set of lines through the origin. In general, using equations (12) and (13), $Sp(2, Z_m)$ is factorized as $Sp(2, Z_{m_1}) \times \cdots \times Sp(2, Z_{m_k})$, that is

$$M(f, g|n, l) = \bigotimes_{j=1}^{k} M(p_j, r_j q_j|\bar{n}_j, l_j) \tag{24}$$

where p_j, q_j, l_j are related l in equation (12) and n_j is related to n in equation(13).

5.1. Factorization in finite geometry

We showcase how prime dimensional finite geometry are embedded in non-prime dimensional finite geometry via divisor function. Using the symplectic matrix defined in equation (15), we factorized lines in finite geometry $G(m)$ as product of its prime finite geomerty with respect to equations (12) and (13). Thus, $Sp(2, Z_m)$ is factorized as $Sp(2, Z_{m_1} \times ... \times Z_{m_k})$,
where $M(f, g|n, l)$ is defined in equation (24) above
Example; $m = 6 \equiv 2 \times 3$; suppose $f = 2, g = 5, n = 1, l = 3$
then $M(2, 5|1, 3)$ is factorized in terms of equation (24) using equations (12) and (13) as;

$$M(2, 5|1, 3) = M(1, 0|1, 1) \otimes M(2, 1|1, 1) \tag{25}$$

More examples are shown in the Table 1 below for $m = 6; m_1 = 2, m_2 = 3, r_1 = 3, r_2 = 2, t_1 = 1, t_2 = 2$,

6. Factorization and partial ordering in finite quantum systems $\mho(m)$ with variables in Z_m

Finite quantum systems $\mho(m)$ with variables in Z_m for m a non prime is factorized as products of its distinct prime $\mho(m_i)$ in this section.

Here, $m_i|m$, Z_m has Z_{m_i} as its subgroup. As a result, we express a finite system $\mho(m)$ has $\mho(m_i)$ as its subsystem.
We use the notations $|X_m;\delta\rangle$ and $|P_m;\delta\rangle$ to denote the positions and momenta states respectively, where $\delta \in Z_m$. The concept

$$F_m = m^{-\frac{1}{2}} \sum_{\delta,N}^{m-1} \omega(\delta N)|X_m;\delta\rangle\langle X_m;N|; \qquad \omega(\delta) = \exp\left(i\frac{2\pi\delta}{m}\right) \tag{26}$$

represents the Fourier transform. The momentum states is obtained by acting the Fourier transform of the position states, that is

$$|P_m;\delta\rangle = F_m|X_m;\delta\rangle. \tag{27}$$

As an illustration suppose $m = 2$, we have

$$F_2 = \frac{1}{\sqrt{2}}[\omega(0)|X_2;0\rangle\langle X_2;0| + \omega(0)|X_2;0\rangle\langle X_2;1| + \omega(0)|X_2;1\rangle\langle X_2;0| + \omega(1)|X_2;1\rangle\langle X_2;1|] \tag{28}$$

This produces

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix} \tag{29}$$

For $m = 3$ we get

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{30}$$

for $m = 6$ we get

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}, \omega^0 = 1 \tag{31}$$

$D(\kappa, \xi)$ represent the displacement operator, it is defined as

$$D(\kappa,\xi) = Z^\kappa X^\xi \omega(-2^{-1}\kappa\xi) \tag{32}$$

where

$$Z^\kappa = \sum_{N=0}^{m-1} \omega(N\kappa)|X_m;N\rangle\langle X_m;N|; \tag{33}$$

$$X^\xi = \sum_{N=0}^{m-1} \omega(-N\kappa)|P_m;N\rangle\langle P_m;\ N| \tag{34}$$

and

$$X^\xi Z^\kappa = Z^\kappa X^\xi \omega(-\kappa\xi); X^m = Z^m = \mathbf{1} \tag{35}$$

The $D(\kappa,\xi)\omega(\mu)$ where $\kappa,\xi,\mu \in Z_m$ forms a group called an Heisenberg-Weyl group.

6.1. Factorization of bases as prime factor bases

In this subsection, we use Chinese Remainders Theorem (CRT) to express a finite quantum system $\mho(m)$ with variables in Z_m as products of its subsystems $\mho(m_i)$ with variables in Z_{m_i}. CRT was used by [10–14] to express a large size finite quantum system $\mho(m)$ with variables in Z_m as products of its component subsystems $\mho(m_1), .., \mho(m_k)$ with variables in Z_{m_k} using equations (12) and (13). Our findings shows an existence of bijection between a large dimension finite Hilbert space $H(m)$ and the tensor products of its components space $H(m_i)$ where $m_1, ..., m_k$ are relatively prime. That is

$$|X_m;\delta\rangle \longleftrightarrow \bigotimes_{j=1}^{k} |X_{p_j};\bar{\delta}_j\rangle, D_m(\kappa,\xi) = \bigotimes_{j=1}^{k} D_{p_j}(\kappa_j,\bar{\xi}_j), H(m) = \bigotimes_{j=1}^{k} H(m_j) \tag{36}$$

(m_j is a prime), where

$$|X_m;\delta\rangle \longleftrightarrow |X_{m_1};\bar{\delta}_1\rangle \otimes ... \otimes |X_{m_k};\bar{\delta}_k\rangle \tag{37}$$

and

$$|P_m;\delta\rangle \longleftrightarrow |P_{m_1};\delta_1\rangle \otimes ... \otimes |P_{m_k};\delta_k\rangle \tag{38}$$

As illustrations, a six dimensional Hilbert space H_6 was factorized as products of two and three dimensional spaces, $H_2 \otimes H_3$, thus using equations (12) and (13) for case $m = 6$, the first bijection is, $5 \longleftrightarrow (1,2)$ and the second bijection is $5 \longleftrightarrow (1,1)$.
Therefore; position states in H_6 is factorized as;

$$|X_6;5\rangle \longleftrightarrow |X_2;1\rangle \otimes |X_3;2\rangle \tag{39}$$

Its momentum states is obtained thus;

$$|P_6;5\rangle \longleftrightarrow |P_2;1\rangle \otimes |P_3;1\rangle \tag{40}$$

Using equation (32), and for $m = 6$, the displacement operator $D(\kappa,\xi)$ is factorized as

$$D_6(3,5) = D_2(1,1) \otimes D_3(0,1) \tag{41}$$

6.2. Embedding of small systems into large systems

We discuss how a small dimensional finite quantum system $\mho(m_i)$ was embedded into a large dimensional finite quantum systems $\mho(m)$ for $m_i|m$. We consider an orthonormal basis $|X_m;\delta\rangle$ where $\delta \in Z_m$.
If $m_i|m$ then $Z_{m_i} \subset Z_m$, it implies $\mho(m) \ni \mho(m_i)$.
Suppose we define a quantum subsystem $\mho(m)$ contained $\mho(m_i)$, an injective map with respect to position state is defined as;

$$\sum_{\delta=0}^{m_i-1} S_\delta \Big|X_{m_i};\delta\rangle \to \sum_{\delta=0}^{m-1} S_\delta \Big|X_m;\frac{m\delta}{m_i}\Big\rangle \tag{42}$$

The above relation in equation (42) is expressed in terms of momentum states as;

$$\sum_{\delta=0}^{m_i-1} T_m \Big|P_{m_i};\delta\Big\rangle \to \sum_{\delta=0}^{m-1} T_\delta \Big|P_m;\frac{m\delta}{m_i}\Big\rangle \tag{43}$$

As illustration, let $m = 6, m_i = 2, 3$; the subgroup of Z_6 are

$$Z_2 = \{0,1\} \text{ and } Z_3 = \{0,1,2\}. \tag{44}$$

We express a finite quantum system $\mho(6)$ as,

$$\mho(6) = |X_6;\delta\rangle = \{|X_6;0\rangle, |X_6;1\rangle, |X_6;2\rangle, |X_6;3\rangle, |X_6;4\rangle, |X_6;5\rangle\}. \tag{45}$$

Its subsystems are

$$|X_3;2\delta\rangle = \{|X_3;0\rangle, |X_3;2\rangle, |X_3;4\rangle\}. \tag{46}$$

$$|X_2;3\delta\rangle = \{|X_2;0\rangle, |X_2;3\rangle\}. \tag{47}$$

$\mho(m_i)$ takes values from Z_{m_i} of Z_m of the variables $\mho(m)$.
It is observed above that equation (46) is embedded in equation (45).

Futhermore, an existence of a $one - one$ map between $\mho(3)$ and $\mho(6)$ is confirmed which implies that the quantum states of $\mho(m_i)$ are embedded into $\mho(m)$ as shown below.

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} \rightarrow \begin{pmatrix} S_0 \\ 0 \\ S_1 \\ 0 \\ S_2 \\ 0 \end{pmatrix} \tag{48}$$

The above relation in equation (42) is expressed in terms of momentum states as;

$$\sum_{\delta=0}^{m_i-1} T_\delta \left| P_{m_i}; \delta \right\rangle \rightarrow \sum_{\delta=0}^{m-1} T_\delta \left| P_m; \frac{m\delta}{m_i} \right\rangle \tag{49}$$

for $m = 6$ and $m_i = 2$ the left hand side (LHS) of equation (49) yields,

$$2^{-\frac{1}{2}} \begin{pmatrix} T_0 + T_1 \\ T_0 + T_1\omega \end{pmatrix} \tag{50}$$

For the right hand side (RHS) of equation (49) we have

$$\sum_{m=0}^{m_i-1} T_\delta \left| P_m; \frac{m\delta}{m_i} \right\rangle = 6^{-\frac{1}{2}} \begin{pmatrix} T_0 + T_1 \\ T_0 + T_1\omega \\ T_0 + T_1 \\ T_0 + T_1\omega \\ T_0 + T_1 \\ T_0 + T_1\omega \end{pmatrix} \tag{51}$$

There exists an injection between equations (50) and (51). This implies that equation (50) is embedded in equation (51) confirming equation (49). That is

$$2^{-\frac{1}{2}} \begin{pmatrix} Tt_0 + T_1 \\ T_0 + T_1\omega \end{pmatrix} \rightarrow 6^{-\frac{1}{2}} \begin{pmatrix} T_0 + T_1 \\ T_0 + T_1\omega \\ T_0 + T_1 \\ T_0 + T_1\omega \\ T_0 + T_1 \\ T_0 + T_1\omega \end{pmatrix} \tag{52}$$

Hence, an existence of partial order relation has been observed in general within the non-prime dimensional finite geometry and finite quantum systems with subgeometries and subsystems as partial order. This thereby demonstrates dualities between geometries and quantum systems.

7. Mutually unbiased bases

Prime dimensional Mutually unbiased bases has been discussed in many works in the past. It is a situation where by the overlap of two orthogonal vectors of finite dimennsion yields $\frac{1}{\sqrt{m}}$. that is

$$|\langle X_{\Delta_i}; \beta | X_{\Delta_j}; \alpha \rangle|^2 = \frac{1}{m}, \ \forall |X_{\Delta_i}; \beta\rangle \in |B(\Delta_\mathbf{i}); \beta\rangle \text{ and } |X_{\Delta_j}; \alpha\rangle \in |B(\Delta_\mathbf{j}); \alpha\rangle, \tag{53}$$

for $\Delta_\mathbf{i} \neq \Delta_\mathbf{j}$.

It was confirmed in [11] that absence of inverse of 2 in even dimensional finite Hilbert space leads to inability to know the number of mutually unbiased bases. As a result we restrict our discussion to finite systems with odd dimension

only.
The displacement operators is defined earlier in equation (32), it forms a representation of Heisenberg-Weyl group. Symplectic transformation has been studied in [10]. It satisfies the conditions;

$$[\mathbb{M}(f,g|m,l)]X_m[\mathbb{M}(f,g|m,l)]^{\dagger} = D(g,f)$$
$$[\mathbb{M}(f,g|m,l)]Z_m[\mathbb{M}(f,g|m,l)]^{\dagger} = D(l,m)$$
$$\mathbb{M}(fl-gm) = 1(mod\ (m)), f,g,m,l \in Z_m \tag{54}$$

In this work $M(fl-gm)$ defined in equation (15) and $\mathbb{M}(fl-gm)$ in equation (54) do not belong to the same representation. The Fourier transform is defined as

$$F_m = \mathbb{M}(0,1|-1,0). \tag{55}$$

The mutually unbiased bases in finite quantum systems with odd dimension thus;
Suppose

$$\Delta = -1 \rightarrow |X_{-1};\alpha\rangle = \mathbb{M}(1,0|0,1)|X_{m_i};\alpha\rangle$$
$$\Delta = 0,...,\mathbf{m_i}-1 \rightarrow |X_{\Delta};\alpha\rangle = \mathbf{M}(0,1|-1,\Delta)|X_{m_i};\alpha\rangle \tag{56}$$

for $\Delta = 0, |X_0;\alpha\rangle = |P_{m_i};\alpha\rangle$.
If we take any two states where these two states are not from the same bases, calculating the modulus of their dot product yields equation (53).
In this case, there exists $\psi(m_i)$ mutually unbiased bases.

$$|B_{\Delta};\alpha\rangle = \{|X_{\Delta};\alpha\rangle\}; \quad \Delta = -1,...,m_i-1 \tag{57}$$

The mutually unbiased bases for prime dimension, $m_i = 3$ is shown below.
Let

$$|B_{-1};\alpha\rangle = \{|X_{-1}(1,0|0,1);\alpha\rangle\}, \alpha \in Z_3 \tag{58}$$

represents the standard bases, here $|X_{-1}(1,0|0,1);\alpha\rangle$ is equivalent to $\mathbb{M}(1,0|0,1)|X_3;\alpha\rangle$.
We obtained the remaining bases by using symplectic transform $|X_{-1};\alpha\rangle$;
$\mathbb{M}(0,1|-1,\Delta)|X_3;\alpha\rangle = |X_{\Delta}(0,1|-1,\Delta);\alpha\rangle$ where $\Delta \in Z_{m_i}$ and $\alpha \in Z_{m_i}$

$$\text{(i) For } \Delta = 0;\ |B_0;\alpha\rangle = \{|X(0)(0,1|-1,0);\alpha\rangle\};$$
$$\text{(ii) For } \Delta = 1;\ |B_1;\alpha\rangle = \{|X(1)(0,1|-1,1);\alpha\rangle\};$$
$$\text{(iii) For } \Delta = 2;\ |B_2;\alpha\rangle = \{|X(2)(0,1|-1,2);\alpha\rangle\}. \tag{59}$$

Taking any two states from distinct bases and calculating the modulus of their dot product yields $m_i^{-\frac{1}{2}}$.

7.1. Factorization of bases and weak mutually unbiased bases ($\mathcal{WMUB}$)

As discussed earlier in Factorization of lines, bases of a non-prime dimesional finite Hilbert space of finite quantum systems $\mho(m)$ is expressed as products of prime dimensional Hilbert space $\mho(m_i)$. This concept had been discussed by many authors [1] and [11–13] was used by Good in [15]. However in our work we mentioned it briefly to showcase the duality in finite geometry in its match in finite quantum systems. Let $\{|B_i;\alpha\rangle\}$ denotes a set of $\mathfrak{g}$ orthonormal bases in the Hilbert spaces $H(m)$ where $\beta \in Z_m$ and $j = 1,2,...,\mathfrak{g}$ is called a weak mutually unbiased bases if.

$$|\langle B_j;\beta|B_i;\alpha\rangle| = m_i^{-\frac{1}{2}} \text{ or } 0,;m_i|m \quad (\mathfrak{i} \neq \mathfrak{j}) \tag{60}$$

Any set of weak mutually unbiased bases in $H(m)$ can be expressed as products of mutually unbiased bases. $|X_{\Delta_1};\bar{\alpha}_1\rangle \otimes ... \otimes |X_{\Delta_k};\bar{\alpha}_k\rangle$ where $\{|X_{\Delta_1};\bar{\alpha}_1\rangle\}$ is a set of mutually unbiased bases in Hilbert subspace $H(m_1)$, $\{|X_{\Delta_2};\bar{\alpha}_2\rangle\}$ is a set of mutually unbiased bases in Hilbert subspace $H(m_2)$, ..., $\{|X_{\Delta_k};\bar{\alpha}_k\}\rangle$ is a set of mutually unbiased bases in Hilbert subspace $H(m_k)$.

This is analogous to expression of a non-prime positive integer as products of its prime factors. Bases in non-prime dimension finite quantum systems is expressed as follows;

$$|X_m;\alpha\rangle = |X_{m_1};\bar{\alpha}_1\rangle \otimes ... \otimes |X_{m_k};\bar{\alpha}_k\rangle, \alpha \in Z_m \quad \bar{\alpha} \in Z_{m_j} \tag{61}$$

As a result from equation (36),the weak mutually unbiased bases is expressed here as

$$|X_{\Delta_1,...,\Delta_k};\bar{\alpha}_1,...,\bar{\alpha}_k\rangle = |\mathcal{X}_{1,\Delta_1};\bar{\alpha}_1\rangle \otimes ... \otimes |X_{k,\Delta_k};\bar{\alpha}_k\rangle \tag{62}$$

where $\bar{\alpha}_j \in Z_{m_j}$ and $-1 \le \alpha_j \le m_j - 1$.
In a special case , if

$$\Delta_1 = ... = \Delta_k = -1, \tag{63}$$

then

$$\begin{aligned} |\mathfrak{X}_{-1,...,-1};\bar{\alpha}_1,...,\bar{\alpha}_k\rangle &= |X_{1,-1};\bar{\alpha}_1\rangle \otimes ... \otimes |X_{k,-1};\bar{\alpha}_k\rangle \\ &= |X_1;\bar{\alpha}_1\rangle \otimes ... \otimes |X_k;\bar{\alpha}_k\rangle \end{aligned} \tag{64}$$

If

$$\Delta_1 = ... = \Delta_k = 0, \tag{65}$$

then

$$\begin{aligned} |X_{0,...,0};\bar{\alpha}_1,...,\bar{\alpha}_k\rangle &= |X_{1,0};\bar{\alpha}_1\rangle \otimes ... \otimes |X_{k,0};\bar{\alpha}_k\rangle \\ &= |P_1;\alpha_1\rangle \otimes ... \otimes |P_k;\alpha_k\rangle \end{aligned} \tag{66}$$

taking the absolute value of the dot product of any two states each belonging to different bases in equations (64) and (66), it satisfies the relation;

$$|\langle \mathfrak{X}_{\Delta_1,...,\Delta_k};\bar{\gamma}_1,...,\bar{\gamma}_{\mathbf{k}}|X_{\Delta_1,...,\Delta_k};\bar{\alpha}_1,...,\bar{\alpha}_k\rangle| = \frac{1}{\sqrt{m_i}} \; or \; 0, \; m_i|m. \tag{67}$$

There exists

$$\psi(m) = \prod_{j=1}^{k}(m_k + 1) \tag{68}$$

maximum number of weak unbiased bases in Hilbert space H_m.

$$|B_{\Delta_1,...,\Delta_\mathbf{k}};\alpha\rangle = \{|X_{\Delta_1,...,\Delta_k};\bar{\alpha}_1,...,\bar{\alpha}_k\rangle\} \tag{69}$$

An existence of the duality between lines in finite geometry and weak mutually unbiased bases was discussed. Table 2 below shows the summary of the duality for line in $G(m)$ and bases in $H(m)$ where $m = 6$.

$|B_{\Delta_1,\Delta_2};\alpha\rangle$ represents bases in a finite Hilbert space of a quantum systems,
$|X_{m_1}(0,1|-1,\Delta_1);\bar{\alpha}_1\rangle$ represents an orthogonal vector in state $\bar{\alpha}_1$, where $\bar{\alpha}_1 \in Z_{m_1}$
$|X_{m_2}(0,1|-1,\Delta_2)$ represents an orthogonal vector in state $\bar{\alpha}_2$, where $\bar{\alpha}_2 \in Z_{m_2}$.

8. Duality between weak mutually unbiased bases in *H*(*m*) and lines in *G*(*m*)

The maximal lines in $G(m)$ corresponds to weak mutually unbiased bases in $H(m)$. The $\psi(m)$ maximal lines in $G(m)$ conforms to $\psi(m)$ weak mutually unbiased bases in $H(m)$. Each maximal lines has m points, also there are m orthogonal vectors in each of *WMUB* in $H(m)$. For $m_i|m$, the subgeometries $G(m_i)$ of $G(m)$ corresponds to the subsystems $\mho(m_i)$ of $\mho(m)$.
There are $\sigma_0(m)$ subgeometries $G(m_i)$ of $G(m)$ and likewise there are $\sigma_0(m)$ subsystems $\mho(m_i)$ of $\mho(m)$.

A phase-space finite geometry G_6 contains:

Table 2. Weak mutually unbiased bases for $H(6) = H(2) \otimes H(3)$.

$\lvert B_{\Delta_1,\Delta_2}; \alpha\rangle$	$\lvert X_{m_1}(0,1\lvert -1,\Delta_1); \bar{\alpha}_1\rangle$	$\lvert X_{m_2}(0,1\lvert -1,\Delta_2); \bar{\alpha}_2\rangle$
$\lvert B_{-1,-1}; \alpha\rangle$	$\lvert X_{2,-1}(1,0\lvert 0,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,-1}(1,0\lvert 0,1); \bar{\alpha}_2\rangle$
$\lvert B_{-1,0}; \alpha\rangle$	$\lvert X_{2,0}(1,0\lvert 0,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,0}(0,1\lvert -1,0); \bar{\alpha}_2\rangle$
$\lvert B_{-1,1}; \alpha\rangle$	$\lvert X_{2,1}(1,0\lvert 0,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,1}(0,1\lvert -1,1); \bar{\alpha}_2\rangle$
$\lvert B_{-1,2}; \alpha\rangle$	$\lvert X_{2,-1}(1,0\lvert 0,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,2}(0,1\lvert -1,2); \bar{\alpha}_2\rangle$
$\lvert B_{0,-1}; \alpha\rangle$	$\lvert X_{2,0}(0,1\lvert -1,0); \bar{\alpha}_1\rangle$	$\lvert X_{3,-1}(1,0\lvert 0,1); \bar{\alpha}_2\rangle$
$\lvert B_{1,-1}; \alpha\rangle$	$\lvert X_{2,1}(0,1\lvert -1,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,-1}(1,0\lvert 0,1); \bar{\alpha}_2\rangle$
$\lvert B_{0,0}; \alpha\rangle$	$\lvert X_{2,0}(0,1\lvert -1,0); \bar{\alpha}_1\rangle$	$\lvert X_{3,0}(0,1\lvert -1,0); \bar{\alpha}_2\rangle$
$\lvert B_{0,1}; \alpha\rangle$	$\lvert X_{2,0}(0,1\lvert -1,0); \bar{\alpha}_1\rangle$	$\lvert X_{3,1}(0,1\lvert -1,1); \bar{\alpha}_2\rangle$
$\lvert B_{0,2}; \alpha\rangle$	$\lvert X_{2,0}(0,1\lvert -1,0); \bar{\alpha}_1\rangle$	$\lvert X_{3,2}(0,1\lvert -1,2); \bar{\alpha}_2\rangle$
$\lvert B_{1,0}; \alpha\rangle$	$\lvert X_{2,1}(0,1\lvert -1,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,0}(0,1\lvert -1,0); \bar{\alpha}_2\rangle$
$\lvert B_{1,1}; \alpha\rangle$	$\lvert X_{2,1}(0,1\lvert -1,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,1}(0,1\lvert -1,1); \bar{\alpha}_2\rangle$
$\lvert B_{1,2}; \alpha\rangle$	$\lvert X_{2,1}(0,1\lvert -1,1); \bar{\alpha}_1\rangle$	$\lvert X_{3,2}(0,1\lvert -1,2); \bar{\alpha}_2\rangle$

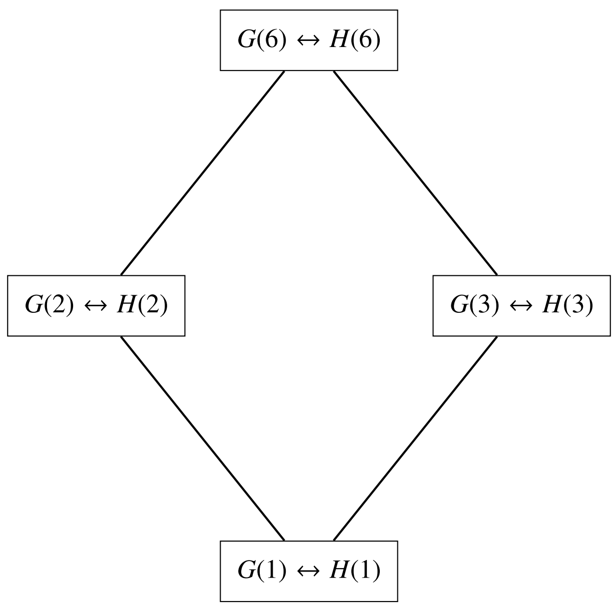

Figure 1. The Hasse diagram showing duality between $G(6)$ and $H(6)$

1. Lines with 6 points subgeometries G_2 and G_3 with lines with 2 and 3 points, respectively. A finite Hilbert space H_6 contains bases each with 6 orthogonal vectors, subspaces H_2 and H_3.

2. A union G_2 and G_3 is isomorphic to subgeometry G_6. A union H_2 and H_3 is isomorphic to subspace of H_6.

3. An intersection of G_2 and G_3 is isomorphic to G_1. Also, an intersection of H_2 and H_3 is isomorphic to subspace H_1.

Hence, from properties (1), (2), and (3) above, the Hasse diagram does not only show a duality but also form a lattice as shown in figure 1.

9. Conclusion

This study pays attention to the existence of lattices in non-near linear finite geometry $G(m)$ and prime geometries $G(m_i)$, as well as the finite quantum system $\Pi(m)$ and its subsystem $\Pi(m_i)$, with subsystems, forming a lattice. More importantly, the results shown in this study demonstrate those important relation which exists between a structure and its substructures both in quantum system and geometry in its phase space.

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