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Numerical Application of Higher Order Linear Block Method for Solving Some Fourth Order Initial Value Problems

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Abstract

The numerical application of higher order linear block method for the direct solution of fourth order initial value problems was proposed using the linear block algorithm, where the methods applied in block form. The method is zero-stabile, consistent and convergent when analyzing the properties of the method. The mathematical example solved using the method is effective, suitable, and acceptable for solving fourth order initial value problems. The method is also compared with existing work when solving similar systems of differential equation and obviously, the method performs better than those in literature and textual shown.

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1. Introduction

Numerical analysis is the field of mathematics which carries a suitable method for solving difficult problems in mathematics and find out suitable material from accessible results which are not stated in tractable forms. Such

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problems initiate for the most part, from real world applications of algebra, geometry, calculus, and they include variables which vary [1].

Mathematical models are abstract models that use mathematical language to explain the behavior of physical phenomena. Someone can reason that, mathematical modeling is an action or procedure that permits a mathematician to adopt the office of a researcher, physicist, chemist, biologist, an ecologist, an economist, a physiologist and so on. Nevertheless, it place of taking research in linear multistep method, a scientist takes a research on mathematical illustrations of the natural processes or physical-world problems (real life problem) [1].

Many natural processes or physical-world problems can be translated into the language of mathematics. The research treats fourth order initial value problem (IVP) of the form:

$$y^{iv}(z) = f(z,\ y,\ y',\ y'',\ y'''),\ y(z_0) = \eta_1,\ y'(z_0) = \eta_2,\ y''(z_0) = \eta_3,\ y'''(z_0) = \eta_4 \tag{1}$$

The differential equation (1) is the foundation for treating multiple of problems ascending from the arenas of sciences, technology, engineering, agriculture, social science, etc.. The system is arranged from first order to higher order problems [2].

Some examiners in literature have treat (1), without following the reduction method because of its setbacks, among others are [2-7].

Solving fourth order initial value problems of ordinary differential equations without reduction process was developed by [3-7], where [4] proposed a six-step zero-stable block method for solving fourth order initial value problems using power series as a basic function and overcome for the setbacks in reduction methods. [5] developed the block algorithm using the linear block method and his method improved on reduction method.

Many research activities along this line have produced numerical schemes that are not enough in handling fourth order problems directly without reducing them to lower order of initial value problems.

The setback associated with such method lies in the fact that they have low order of accuracy, time constraint and hence the performances of their methods are not good enough.

Some researchers have treated higher order ordinary differential equation (1) directly without going through the process of reduction because it is difficult, among others are [1-4, 8-11]. We notice that direct solutions is competent than reduction method in term of accuracy. The essence of this work is to avoid the inherent drawbacks associated with reduction methods and to improve accuracy.

2. Mathematical Algorithm of the Methodology

The method for solving (1) is derived using Block Algorithm. Prepositions one and two showed the stages to develop the scheme. The method leads the model of considering the general form of the algorithm while following one by one to yield the expected method for solving fourth order initial value problem [1].

Preposition one is used to yield the block algorithm using:

$$y_{n+\xi} = \sum_{j=0}^{3} \frac{(\xi h)^j}{j!} y_n^{(j)} + \sum_{j=0}^{5} \left(\delta_{i\xi} f_{n+j}\right),\ \xi = r,\ s,\ u,\ v,\ w,\ 1 \tag{2}$$

Preposition two is used to yield the first, second and third derivative of the algorithm from

$$y_{n+\xi}^{(g)} = \sum_{j=0}^{(3-g)} \frac{(\xi h)}{j!} + \sum_{j=0}^{5} \Omega_{\xi j g} f_{n+j},\ g = 1_{(\xi=r,\ s,\ u,\ v,\ w,\ 1)},\ g = 2_{(\xi=r,\ s,\ u,\ v,\ w,\ 1)},\ g = 3_{(\xi=r,\ s,\ u,\ v,\ w,\ 1)} \tag{3}$$

where

$$\delta_{ij} = A^{-1} Z \ and \ \Omega_{\xi jg} = A^{-1} Y$$

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & \frac{(r)^1}{1!} & \frac{(s)^1}{1!} & \frac{(u)^1}{1!} & \frac{(v)^1}{1!} & \frac{(w)^1}{1!} & \frac{(1)^1}{1!}\\ 0 & \frac{(r)^2}{2!} & \frac{(s)^2}{2!} & \frac{(u)^2}{2!} & \frac{(v)^2}{2!} & \frac{(w)^2}{2!} & \frac{(1)^2}{2!}\\ 0 & \frac{(r)^3}{3!} & \frac{(s)^3}{3!} & \frac{(u)^3}{3!} & \frac{(v)^3}{3!} & \frac{(w)^3}{3!} & \frac{(1)^3}{3!}\\ 0 & \frac{(r)^4}{4!} & \frac{(s)^4}{4!} & \frac{(u)^4}{4!} & \frac{(v)^4}{4!} & \frac{(w)^4}{4!} & \frac{(1)^4}{4!}\\ 0 & \frac{(r)^5}{5!} & \frac{(s)^5}{5!} & \frac{(u)^5}{5!} & \frac{(v)^5}{5!} & \frac{(w)^5}{5!} & \frac{(1)^5}{5!}\\ 0 & \frac{(r)^6}{6!} & \frac{(s)^6}{6!} & \frac{(u)^6}{6!} & \frac{(v)^6}{6!} & \frac{(w)^6}{6!} & \frac{(1)^6}{6!} \end{pmatrix},\ X=\begin{pmatrix} \frac{(\xi)^4}{4\,!}\\ \frac{(\xi)^5}{5\,!}\\ \frac{(\xi)^6}{6\,!}\\ \frac{(\xi)^7}{7\,!}\\ \frac{(\xi)^8}{8\,!}\\ \frac{(\xi)^9}{9\,!}\\ \frac{(\xi)^{10}}{10\,!} \end{pmatrix},\ Y=\begin{pmatrix} \frac{(\xi)^{4-q}}{(4-q)\,!}\\ \frac{(\xi)^{5-q}}{(5-q)\,!}\\ \frac{(\xi)^{6-q}}{(6-q)\,!}\\ \frac{(\xi)^{7-q}}{(7-q)\,!}\\ \frac{(\xi)^{8-q}}{(8-q)\,!}\\ \frac{(\xi)^{9-q}}{(9-q)\,!}\\ \frac{(\xi)^{10-q}}{(10-q)\,!} \end{pmatrix}$$

In order to find the unknown values $r,\ s,\ u,\ v,\ w$, for$b_i's$, we solve (3) using Gaussian elimination method to yield a continuous hybrid linear multistep method of the form.

$$p\ (x)\ =\sum_{j=r,\ s,\ u,\ v}\psi_j y_{n+j}+h^4\left[\sum_{i=0}^{1}\varphi_j f_{n+j}+\varphi_m f_{n+m}\right],\ m=0,\ r,\ s,\ u,\ v,\ w,\ 1 \tag{4}$$

On solving the above equation to yield the coefficients of the polynomial $\varphi_m,\ m=0,\ r,\ s,\ u,\ v,\ w,\ 1$

By making the Substitution $x=x_m+th$, the polynomial in eqn (2) takes the form.

$$\begin{aligned} p\left(x_m+th\right) &= a_0y_m+a_ry_{m+r}+a_sy_{m+s}a_uy_{m+u}+\\ &\quad h^4\left(\beta_0f_m+\beta_rf_{m+r}+\beta_sf_{m+s}+\beta_uf_{m+u}+\beta_vf_{m+v}+\beta_wf_{m+w}+\beta_1f_{m+1}\right) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \varphi_0 &= 1\\ \varphi_r &= \xi\\ \varphi_s &= \tfrac{1}{2}\xi^2\\ \varphi_u &= \tfrac{1}{6}\xi^3 \end{aligned}$$

$$\varphi_0=\frac{1}{15120rsuvw}\xi^4\left(\begin{array}{l} 3\xi^6-5\xi^5+9r\xi^4-5r\xi^5+9s\xi^4-5s\xi^5+9u\xi^4-5u\xi^5+9v\xi^4-5v\xi^5\\ +9w\xi^4-5w\xi^5-18rs\xi^3+9rs\xi^4-18ru\xi^3+9ru\xi^4-18rv\xi^4-18su\xi^3\\ +9rv\xi^4+9su\xi^4-18rw\xi^3-18sv\xi^3+9rw\xi^4+9uw\xi^4-18sw\xi^3+9sw\xi^4\\ -18uv\xi^3+9uv\xi^4-18uw\xi^3+9uw\xi^4-18vw\xi^3+9vw\xi^4+42rsu\xi^2-18rsu\xi^3\\ +42rsv\xi^2-18rsv\xi^3+42rsw\xi^2-18rsw\xi^3+42ruv\xi^2-18ruv\xi^3+42ruw\xi^2\\ +42suv\xi^2-18ruw\xi^3-18suv\xi^3+42rvw\xi^2+42suw\xi^2-18rvw\xi^3-18suw\xi^3\\ +42svw\xi^2-18svw\xi^3+42uvw\xi^2-18uvw\xi^3-126rsuv\xi-126rsuw\xi-126rsvw\xi\\ -126ruvw\xi-126suvw\xi+42rsuv\xi^2+42rsuw\xi^2+42rsvw\xi^2+42ruvw\xi^2\\ +42suvw\xi^2+630rsuvw-126rsuvw\xi \end{array}\right)$$

$$\varphi_r=\frac{1}{15120r}\frac{\xi^5\left(\begin{array}{l} 3\xi^5-5\xi^4+9s\xi^3-5s\xi^4+9u\xi^3-5u\xi^4+9v\xi^3-5v\xi^4+9w\xi^3-5w\xi^4-18su\xi^2+9su\xi^3\\ -18sv\xi^2+9sv\xi^3-18sw\xi^2+9sw\xi^3-18uv\xi^2+9uv\xi^3-18uw\xi^2+9uw\xi^3-18vw\xi^2\\ +9vw\xi^3+42suv\xi+42suw\xi+42svw\xi+42uvw\xi-18suv\xi^2-18suw\xi^2-18svw\xi^2\\ -18uvw\xi^2-126suvw\xi^2-42suvw\xi \end{array}\right)}{(r-s)(r-u)(r-v)(r-w)(r-1)}$$

$$\varphi_s=-\frac{1}{15120s}\frac{\xi^5\left(\begin{array}{l} 3\xi^5-5\xi^4+9r\xi^3-5r\xi^4+9u\xi^3-5u\xi^4+9v\xi^3-5v\xi^4+9w\xi^3-5w\xi^4-18ru\xi^2+9ru\xi^3\\ -18rv\xi^2+9rv\xi^3-18rw\xi^2+9rw\xi^3-18uv\xi^2+9uv\xi^3-18uw\xi^2+9uw\xi^3-18vw\xi^2\\ +9vw\xi^3+42ruv\xi+42ruw\xi+42rvw\xi+42uvw\xi-18ruv\xi^2-18ruw\xi^2-18rvw\xi^2\\ -18uvw\xi^2-126ruvw\xi^2-42ruvw\xi \end{array}\right)}{(r-s)(s-u)(s-v)(s-w)(s-1)}$$

$$\varphi_u=\frac{1}{15120u}\frac{\xi^5\left(\begin{array}{l} 3\xi^5-5\xi^4+9r\xi^3-5r\xi^4+9s\xi^3-5s\xi^4+9v\xi^3-5v\xi^4+9w\xi^3-5w\xi^4-18rs\xi^2+9rs\xi^3\\ -18rv\xi^2+9rv\xi^3-18rw\xi^2+9rw\xi^3-18sv\xi^2+9sv\xi^3-18sw\xi^2+9sw\xi^3-18vw\xi^2\\ +9vw\xi^3+42rsv\xi+42rsw\xi+42rvw\xi+42svw\xi-18rsv\xi^2-18rsw\xi^2-18rvw\xi^2\\ -18svw\xi^2-126rsvw\xi^2-42rsvw\xi \end{array}\right)}{(r-u)(s-u)(u-v)(u-w)(u-1)}$$

Table 1. Coefficient of $\phi'_j s$ and $\varphi'_j s$for the first derivative of (4) which evaluated at all points to give

t_n	y'_n	y''_n	y'''_n	f_n	$f_{n+\frac{1}{6}}$	$f_{n+\frac{1}{3}}$	$f_{n+\frac{1}{2}}$	$f_{n+\frac{2}{3}}$	$f_{n+\frac{5}{6}}$	f_{n+1}
$t_{n+\frac{1}{6}}$	1	$\frac{1}{6}$	$\frac{1}{272}$	$\frac{343801}{783820800}$	$\frac{6031}{9331200}$	$-\frac{32981}{52254720}$	$\frac{5177}{9797760}$	$-\frac{15107}{52254720}$	$\frac{5947}{65318400}$	$-\frac{9809}{783820800}$
$t_{n+\frac{1}{3}}$	1	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{6887}{3061800}$	$\frac{1499}{255150}$	$-\frac{233}{58320}$	$\frac{52}{15309}$	$-\frac{379}{204120}$	$\frac{149}{255150}$	$-\frac{491}{6123600}$
$t_{n+\frac{1}{2}}$	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1959}{358400}$	$\frac{1599}{89600}$	$-\frac{537}{71680}$	$\frac{1}{120}$	$-\frac{327}{71680}$	$\frac{129}{89600}$	$-\frac{71}{358400}$
$t_{n+\frac{2}{3}}$	1	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{3863}{382725}$	$\frac{4664}{127575}$	$-\frac{226}{25515}$	$\frac{272}{15309}$	$-\frac{31}{3645}$	$\frac{344}{127575}$	$-\frac{142}{382725}$
$t_{n+\frac{5}{6}}$	1	$\frac{5}{6}$	$\frac{25}{75}$	$\frac{505625}{31352832}$	$\frac{162125}{2612736}$	$-\frac{85625}{10450944}$	$\frac{66875}{1959552}$	$-\frac{119375}{10450944}$	$\frac{1625}{373248}$	$-\frac{18625}{31352832}$
t_{n+1}	1	1	$\frac{1}{2}$	$\frac{33}{1400}$	$\frac{33}{350}$	$-\frac{3}{560}$	$\frac{2}{35}$	$-\frac{3}{280}$	$\frac{33}{350}$	$-\frac{1}{1200}$

Table 2. Coefficient of $\phi''_j s$ and $\varphi''_j s$ for the second derivative of (4) which evaluated at all points to give

t_n	y'_n	y''_n	y'''_n	f_n	$f_{n+\frac{1}{6}}$	$f_{n+\frac{1}{3}}$	$f_{n+\frac{1}{2}}$	$f_{n+\frac{2}{3}}$	$f_{n+\frac{5}{6}}$	f_{n+1}
$t_{n+\frac{1}{6}}$	0	1	$\frac{1}{6}$	$\frac{28549}{4354560}$	$\frac{275}{20736}$	$-\frac{5717}{483840}$	$\frac{10621}{1088640}$	$-\frac{7703}{1451520}$	$\frac{403}{241920}$	$-\frac{199}{870912}$
$t_{n+\frac{1}{3}}$	0	1	$\frac{1}{3}$	$\frac{1027}{68040}$	$\frac{97}{1890}$	$-\frac{2}{81}$	$\frac{197}{8505}$	$-\frac{97}{7560}$	$\frac{23}{5670}$	$-\frac{19}{34020}$
$t_{n+\frac{1}{2}}$	0	1	$\frac{1}{2}$	$\frac{253}{10752}$	$\frac{165}{1792}$	$-\frac{267}{17920}$	$\frac{5}{128}$	$-\frac{363}{17920}$	$\frac{57}{8960}$	$-\frac{47}{53760}$
$t_{n+\frac{2}{3}}$	0	1	$\frac{2}{3}$	$\frac{272}{8505}$	$\frac{375}{2835}$	$-\frac{2}{945}$	$\frac{272}{15309}$	$-\frac{2}{81}$	$\frac{8}{945}$	$-\frac{2}{1701}$
$t_{n+\frac{5}{6}}$	0	1	$\frac{5}{6}$	$\frac{35225}{870912}$	$\frac{8375}{48384}$	$-\frac{3125}{290304}$	$\frac{25625}{217728}$	$-\frac{625}{96768}$	$\frac{275}{20736}$	$-\frac{1375}{870912}$
t_{n+1}	0	1	1	$\frac{41}{840}$	$\frac{3}{14}$	$\frac{3}{140}$	$\frac{17}{105}$	$\frac{3}{280}$	$\frac{3}{70}$	0

Table 3. Coefficient of $\phi'''_j s$ and $\varphi'''_j s$ for the third derivative of (4) which evaluated at all points to give

t_n	y'_n	y''_n	y'''_n	f_n	$f_{n+\frac{1}{6}}$	$f_{n+\frac{1}{3}}$	$f_{n+\frac{1}{2}}$	$f_{n+\frac{2}{3}}$	$f_{n+\frac{5}{6}}$	f_{n+1}
$t_{n+\frac{1}{6}}$	0	0	1	$\frac{19087}{362880}$	$\frac{2713}{15120}$	$-\frac{15487}{120960}$	$\frac{293}{2835}$	$-\frac{6737}{120960}$	$\frac{263}{15120}$	$-\frac{863}{362880}$
$t_{n+\frac{1}{3}}$	0	0	1	$\frac{1139}{22680}$	$\frac{47}{189}$	$-\frac{11}{7560}$	$\frac{166}{2835}$	$-\frac{269}{7560}$	$\frac{11}{945}$	$-\frac{37}{22680}$
$t_{n+\frac{1}{2}}$	0	0	1	$\frac{137}{2688}$	$\frac{27}{112}$	$-\frac{387}{4480}$	$\frac{17}{105}$	$-\frac{243}{4480}$	$\frac{9}{4480}$	$-\frac{29}{13440}$
$t_{n+\frac{2}{3}}$	0	0	1	$\frac{143}{2835}$	$\frac{232}{945}$	$\frac{64}{945}$	$\frac{752}{2835}$	$\frac{29}{945}$	$\frac{8}{945}$	$-\frac{4}{945}$
$t_{n+\frac{5}{6}}$	0	0	1	$\frac{3715}{72576}$	$\frac{725}{3024}$	$\frac{2125}{24192}$	$\frac{125}{2125}$	$\frac{3875}{24192}$	$\frac{235}{3024}$	$-\frac{275}{72576}$
t_{n+1}	0	0	1	$\frac{41}{840}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{34}{105}$	$\frac{9}{280}$	$\frac{9}{35}$	$\frac{41}{840}$

$$\varphi_v = \frac{1}{15120v}\frac{\xi^5\left(\begin{array}{l} 3\xi^5-5\xi^4+9r\xi^3-5r\xi^4+9s\xi^3-5s\xi^4+9u\xi^3-5u\xi^4+9w\xi^3-5w\xi^4-18rs\xi^2+9rs\xi^3 \\ -18ru\xi^2+9ru\xi^3-18rw\xi^2+9rw\xi^3-18su\xi^2+9su\xi^3-18sw\xi^2+9sw\xi^3-18uw\xi^2 \\ +9uw\xi^3+42rsu\xi+42rsw\xi+42ruw\xi+42suw\xi-18rsu\xi^2-18rsw\xi^2-18ruw\xi^2 \\ -18suw\xi^2-126rsuw\xi^2-42rsuw\xi \end{array}\right)}{(r-v)(s-v)(u-v)(v-w)(v-1)}$$

$$\varphi_w = \frac{1}{15120w}\frac{\xi^5\left(\begin{array}{l} 3\xi^5-5\xi^4+9r\xi^3-5r\xi^4+9s\xi^3-5s\xi^4+9u\xi^3-5u\xi^4+9v\xi^3-5v\xi^4-18rs\xi^2+9rs\xi^3 \\ -18ru\xi^2+9ru\xi^3-18rv\xi^2+9rv\xi^3-18su\xi^2+9su\xi^3-18sv\xi^2+9sv\xi^3-18uv\xi^2 \\ +9uv\xi^3+42rsu\xi+42rsv\xi+42ruv\xi+42suv\xi-18rsu\xi^2-18rsv\xi^2-18ruv\xi^2 \\ -18suv\xi^2-126rsuv\xi^2-42rsuv\xi \end{array}\right)}{(r-w)(s-w)(u-w)(v-w)(w-1)}$$

$$\varphi_1 = -\frac{1}{15120}\frac{\xi^5\left(\begin{array}{l} 3\xi^5-5r\xi^4-5s\xi^4-5u\xi^4-5v\xi^4-5w\xi^4+9rs\xi^3 \\ +9ru\xi^3+9rv\xi^3+9su\xi^3+9rw\xi^3+9sv\xi^3+9sw\xi^3+9uv\xi^3+9uw\xi^3+9vw\xi^3 \\ -18rsu\xi^2-18rsv\xi^2-18rsw\xi^2-18ruv\xi^2-18suv\xi^2-18rvw\xi^2-18suw\xi^2-18svw\xi^2 \\ -18uvw\xi^2+42rsuv\xi+42rsuw\xi+42rsvw\xi+42ruvw\xi+42suvw\xi-126rsuvw\xi \end{array}\right)}{(r-1)(s-1)(u-1)(v-1)(w-1)}$$

Evaluating the first, second and third derivative of (4) at all points we obtain equations as shown in Table 1, 2 and 3.

3. Analyzing the Properties of the Block Scheme

The analyses of the method are analyzed in this section [7].

3.1. Order of the Method

The linear block method (3) is of uniform order four together with error constants

$$C_8 = \left[\frac{275}{40633270272}, \frac{1}{198404640}, \frac{1}{16721504}, \frac{1}{198404640}, \frac{275}{40633270272}, \frac{1}{223948800}\right]^T.$$

3.2. Consistency

The linear block method (3) is consistent according to [6] if all the following situations are fulfilled.

1. The order of the scheme must be greater than or equal to one i.e.,$(p \geq 1)$
2. The linear multistep method $\sum_{j=0}^{k} \alpha_j = 0$and $\alpha_j = 0$
3. $p(z) = p'(z) = 0$ for $z = 1$
4. $p''(z) = 2!\sigma(z)$ for $z = 1$
5. $p'''(z) = 3!\sigma(z)$ for $z = 1$
6. $p''''(z) = 4!\sigma(z)$ for $z = 1$

Therefore, the linear block scheme (3) is consistent.

3.3. Zero stability of our method

The linear block scheme (3) is zero-stable sine there are no root of the first characteristic polynomial $\rho(z) is$ taking a modulus greater than one and all root of modulus one is simple.

That is$\rho(z) = \det\left[zA^0 - A'\right] = 0.$

Mathematically,

$$\det\left[zA^0 - A'\right] = \left\| \begin{bmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\| = 0$$

Hence, the scheme is zero-stable since the roots of the above matrix are $0, 0, 0, 0, 0$ with max$z = 1$.

3.4. Convergence

The essential and appropriate conditions for the block algorithm to be convergent are that they must be consistent and zero-stable. Hence, the block algorithm is convergent since it is both consistent and zero-stable.

3.5. Linear Stability

The concept of A-stability according to [8] is discussed by applying the test equation.

$$y^{(k)} = \lambda^{(k)} y \tag{6}$$

To yield

$$Y_m = \mu(z) Y_{m-1}, \; z = \lambda h \tag{7}$$

Where $\mu(z)$ is the amplification matrix given by

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^4\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^4\eta^{(1)}\right) \tag{8}$$

The matrix $\mu(z)$ has Eigen values $(0, \; 0, \; \cdots, \; \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for two-step fourth derivative method with five offgrid equidistant hybrid points is given by:

$$\xi = -\frac{\begin{pmatrix} 87621165z^6 - 3211626061z^5 + 49980252973z^4 - 585330956910z^3 + 404735811210z^2 \\ -1818121344000z + 35554111488000 \end{pmatrix}}{\begin{pmatrix} 10886400z^6 - 3200601600z^5 + 53038540800z^4 - 576108288000z^3 + 4115059200000z^2 \\ -17777055744000z + 35554111488000 \end{pmatrix}} \tag{9}$$

4. Numerical Experiment of the method

We consider some mathematical examples in order to validate the method. All calculations and programs are carried out using Maple 18 software. Three different examples were considered, and the results are compared with the existing methods. In particular, the methods proposed by [6, 9, 12, 13] were considered.

System 1: Consider the fourth order initial value problem solved by [12, 13]

$$y^{iv} = \sin z + \cos z, \; y'''(0) = \; 7, \; y''(0) = 0, \; y'(0) = -1, \; y(0) = 0,$$

with exact solution given by

$$y(z) = -\sin z + \cos z + z^3 - 1$$

System 2: Consider the fourth order initial value problem solved by [9. 12]

$$y^{iv} = -\left[3y'' + (2 + \alpha\cos(\mathrm{X} z))\right], y(0) = 1, \; y'(0) = \; y''(0) = \; y'''(0) = 0$$

Where $\alpha = 0$for the existence of the theoretical solution,

$$y(z) = 2\cos z - \cos\left(z\sqrt{2}\right)$$

System 3: Consider the fourth order initial value problem solved by [6]

$$y^{iv} - 4y''' + 6y'' + y' + y = 0, \; y(0) = 0, \; y'(0) = \; 1, \; y''(0) = 0, \; y'''(0) = 1$$

with exact solution given by

$$y(z) = z\exp(z) - z^2\exp(z) + \frac{2}{3}z^3\exp(z)$$

Table 4. Relationship of block algorithm with [12, 13] for solving system 1

z	Exact Solution	Commutated Solution	Error in new Method	Error in [13]	Error in [12] block method	Error in [12] PC method
0.103125	-0.10715827398597420496	-0.10715827398597420499	3.0000e-20	5.8350e-18	5.8350e-18	9.9903e-13
0.20625	-0.21721138439467978352	-0.21721138439467978365	1.3000e-19	4.6708e-17	4.6712e-17	2.0098e-12
0.309375	-0.32232814736370974720	-0.32232814736370974757	3.7000e-19	5.2467e-17	4.3748e-16	1.2049e-11
0.4125	-0.41458999434101824709	-0.41458999434101824794	8.5000e-19	9.3430e-17	2.3340e-16	3.0118e-11
0.515625	-0.48600519206074004344	-0.48600519206074004510	1.6600e-18	9.9220e-17	2.3920e-16	6.3035e-11
0.61875	-0.52852383991927116782	-0.52852383991927117074	2.9200e-18	1.4019e-16	2.8020e-16	1.2285e-10
0.721875	-0.53405348539932914285	-0.53405348539932914753	4.6800e-18	1.4613e-16	6.7177e-16	2.2019e-10
0.825	-0.49447519162273866343	-0.49447519162273867048	7.0500e-18	1.8712e-16	4.6706e-16	3.5856e-10
0.928125	-0.40165988630866535270	-0.40165988630866536284	1.0140e-17	1.9324e-16	5.1408e-16	6.6871e-10
1.03125	-0.24748481842399048516	-0.24748481842399049918	1.4020e-17	5.8350e-18	5.8350e-18	9.9903e-13

Source [12, 13]

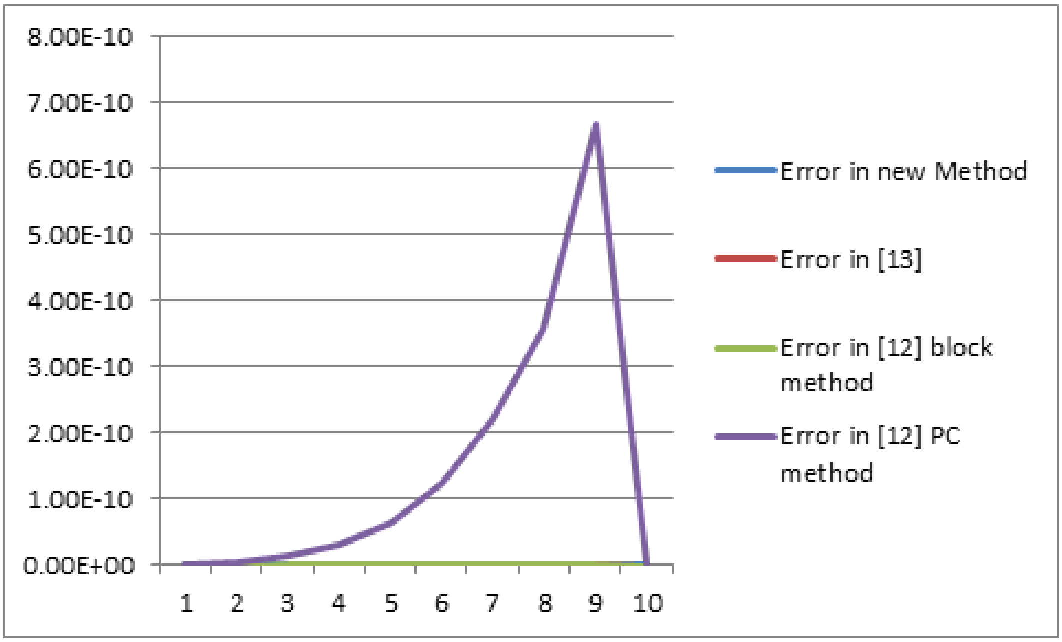

Figure 1. Graphical error of our method with [12, 13] when solving system 1

Table 5. Relationship of block algorithm with [9, 12] for solving system 2

z	Exact Solution	Commutated Solution	Error in new Method	Error in [12] block method	Error in [12] PC method	Error in [9]
0.003125	0.99999999999205272181	0.99999999999205272154	9.9999e-19	6.6858e-13	5.6858e-10	4.2618e-14
0.003125	0.99999999987284392123	0.99999999987284391190	9.3300e-18	1.4585e-11	1.7677e-10	7.3148e-13
0.009375	0.99999999935627549414	0.99999999935627543104	6.3100e-17	1.0830e-10	5.9099e-09	1.6253e-12
0.001250	0.99999999796552658062	0.99999999796552635169	2.2893e-16	3.9178e-10	5.7677e-09	6.5690e-12
0.015625	0.99999999503306753347	0.99999999503306692893	6.0454e-16	1.0252e-09	1.1002e-08	2.8492e-12
0.018750	0.99999998970067947569	0.99999998970067815693	1.3188e-15	2.2173e-09	6.8988e-08	4.2315e-12
0.021875	0.99999998091947944412	0.99999998091947691252	2.5316e-15	4.2261e-09	4.6364e-08	1.2469e-11
0.025000	0.99999996744995111889	0.99999996744994668532	4.4336e-15	7.3580e-09	5.7877e-07	5.1461e-11
0.028125	0.99999994786198113959	0.99999994786197389321	7.2464e-15	1.1969e-09	2.2458e-07	4.1331e-11
0.031250	0.99999992053490100516	0.99999992053488978277	1.1222e-14	1.8463e-08	2.8463e-07	3.7654e-11

Source [9, 12]

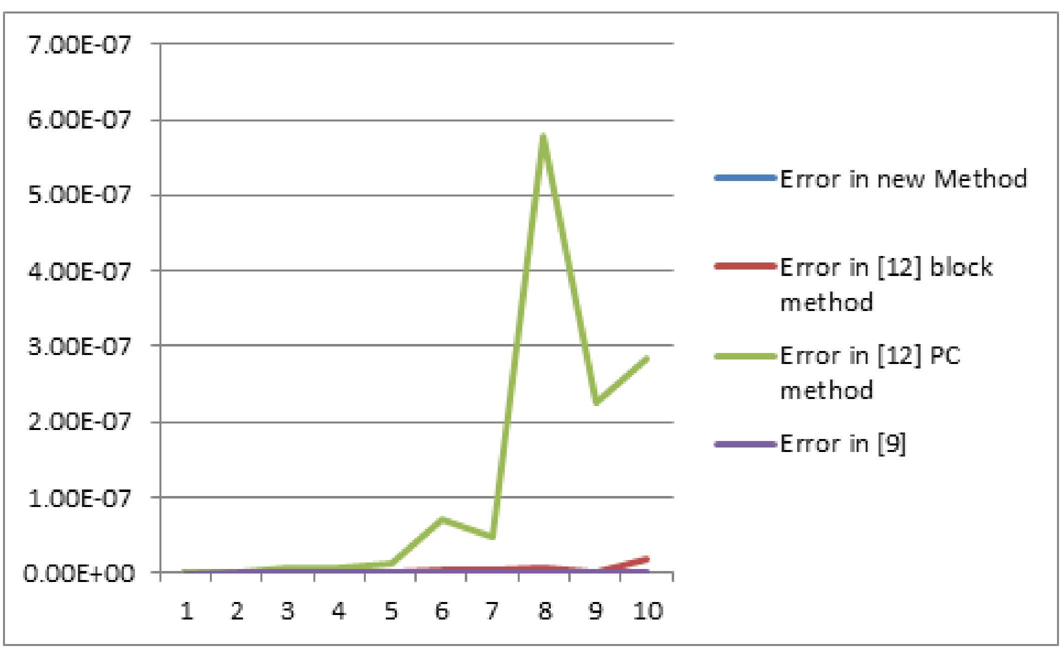

Figure 2. Graphical error of our method with [9, 12] when solving system 2

Table 6. Relationship of block algorithm with [6] for solving system 3

z	Exact Solution	Commutated Solution	Error in new Method	Error in [6]
0.01	0.01000017002091358751	0.01000017002091131988	2.2676e-15	4.0458e-10
0.02	0.02000138733841965720	0.02000138733833788991	8.1767e-14	5.1407e-09
0.03	0.03000477512022127913	0.03000477511965850380	5.6278e-13	2.3703e-08
0.04	0.04001154165741984392	0.04001154165535324981	2.0666e-12	7.1720e-08
0.05	0.05002298300809129940	0.05002298300255914389	5.5322e-12	1.7145e-07
0.06	0.06004048570009828723	0.06004048568786081448	1.2238e-11	3.5227e-07
0.07	0.07006552949425159098	0.07006552947042962362	2.3822e-11	1.1138e-06
0.08	0.08009969020895303375	0.08009969016664333546	4.2310e-11	———
0.09	0.09014464260747098193	0.09014464253733746055	7.0134e-11	1.7939e-06
0.10	0.10020216334901892573	0.10020216323885871798	1.1016e-10	2.7553e-06

Source [6]

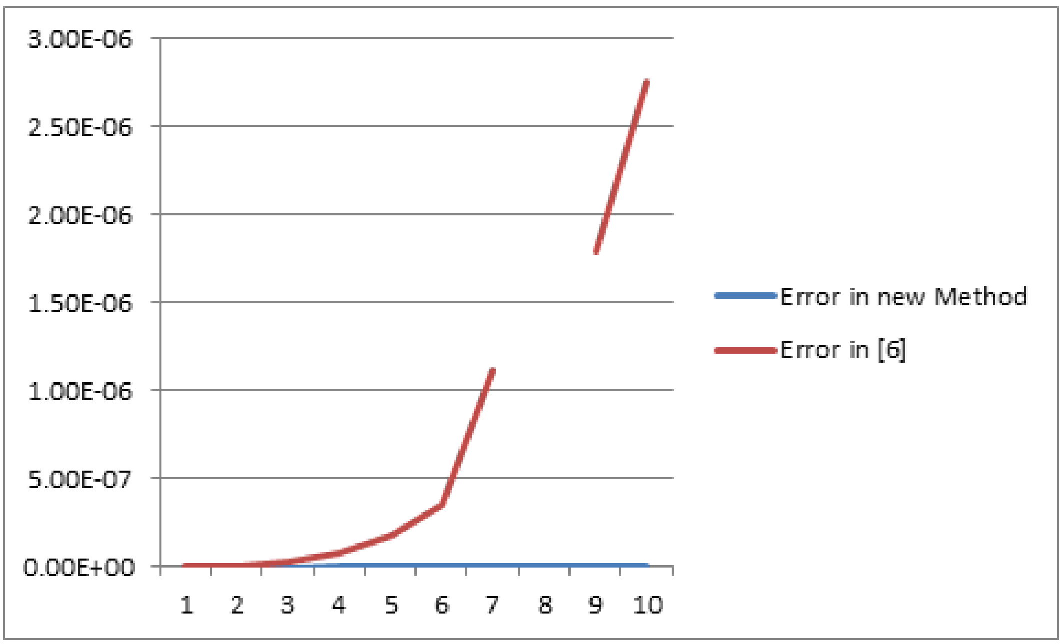

Figure 3. Graphical error of our method with [6] when solving system 3

5. Discussion of Result and Conclusion

The numerical application of higher order linear block method for the direct solution of fourth order initial value problems was proposed using the linear block algorithm. The method was obtained from two different prepositions. The methods are applied in block form. When analyzing the properties of the method, it was found to be zero-stable, consistent, and convergent. Also, the order and error constant are established.

The results obtained using the new method is obviously shown efficient and adequate than that of [6, 9, 12, 13] when solving similar examples.

This research work has made available procedures for developing linear multistep methods using the linear block algorithm. It has also produced numerical methods that are capable of treating (1) without reduction process.

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