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A Modified Two Parameter Estimator with Different Forms of Biasing Parameters in the Linear Regression Model

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Abstract

Despite its common usage in estimating the linear regression model parameters, the ordinary least squares estimator often suffers a breakdown when two or more predictor variables are strongly correlated. This study proposes an alternative estimator to the OLS and other existing ridge-type estimators to tackle the problem of correlated regressors (multicollinearity). The properties of the proposed estimator were derived, and six forms of biasing parameter k (generalized, median, mid-range, arithmetic, harmonic and geometric means) were used in the proposed estimator to compare its performance with five other existing estimators through a simulation study. The proposed estimator dominated existing estimators when the mid-range, arithmetic mean, and median versions of k were used. However, the proposed estimator did not perform well when the generalized, harmonic, and geometric mean versions were used.

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1. Introduction

The Ordinary Least Squared Estimator (OLSE) is the most widely used parameter estimation method in the classical linear regression model. This is because it performs best when some assumptions are not violated. One of these

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assumptions is that the predictor variables are not strongly related. When predictor variables are related, it results in a problem known as multicollinearity. Although OLSE is the best linear unbiased estimator, it produces estimates that deviate greatly from the true parameters and large variances when there are correlated regressors [1]. OLSE is highly likely to be incorrect and unsatisfactory in the presence of multicollinearity since the prediction vector will not be orthogonal [2].

Different estimators have been proposed over the years to handle the problem of multicollinearity. While some of these estimators are one-parameter, others are two-parameter estimators. Examples of one-parameter estimators are the Stein estimator [3], the principal component estimator [4], the ordinary ridge regression estimator by Hoerl and Kennard [2], and the modified ridge regression by Swindel [5], the contraction estimator [6], Liu estimator [7], and Kibria and Lukman [8].

Some two-parameter estimators that have been proposed in works of literature include Ozkale and Kaçiranlar [9]; Yang and Chang [10]; Dorugade [11], Roozbeh [12]; Ayinde *et al.* [13]; Lukman *et al.* [14], Dawoud and Kibria [15], Oladapo *et al.* [16] and more recently Owolabi *et al*. [17-18]. This article proposes a new estimator with prior information, and its properties are derived. Using the mean square error matrix (MSEM) criterion, the performance of the proposed estimator over the OLS estimator, ridge estimator proposed by Hoerl and Kennard [2], Liu estimator [7], Dorugade estimator [11], Kibria-Lukman (K.L.) estimator [8] and the estimator proposed by Ozkale and Kaçiranlar [9] is discussed. The Ridge regression uses the biased parameter *k,* which produces an estimation with a smaller Mean Square Error [19]. However, the performance of biased estimators has been discovered to be affected by choice of biasing parameter *k*. This work, therefore, considers different forms of biasing parameters in the proposed estimator to ascertain the best form of *k*.

In this study, a new two-parameter estimator with prior information is developed, and its properties are derived and compared with some existing estimators using the mean square error criterion. Simulation studies are conducted to compare the performance of the proposed estimator with some existing estimators.

2. Existing and Proposed Estimators

2.1. Some Alternative Ridge Estimators to OLSE

Consider the general linear regression model:

$$y = X\beta + \varepsilon, \varepsilon \sim \left(0, \sigma^2 I\right) \tag{1}$$

where $y$is a $n \times 1$ vector of the response variable, $X$is a known $n \times p$full-rank matrix of predictor variables, β is a $p \times 1$vector of unknown regression parameters to be estimated, and $\varepsilon$is $n \times 1$vector of random error.

The ordinary least square estimator (OLSE) of β in (1) can be defined as:

$$\hat{\beta}_{OLS} = (W)^{-1} X'y \tag{2}$$

where $W = X'X$is the design matrix.

Equation (1) can be expressed in a canonical form as:

$$y = Z\alpha + \varepsilon \tag{3}$$

where $Z = XQ$, $\alpha = Q'\beta$and Q is the orthogonal matrix whose columns constitute the eigenvectors $X'X$. Then, $Z'Z = Q'X'XQ = \Lambda = diag\ \left(\lambda_1, ..., \lambda_p\right)$,

where $\lambda_1 \geq \lambda_2 \geq ...\lambda_p > 0$ are the ordered eigen values $X'X$. The ordinary least square estimator (OLSE) of $\beta$and the mean squared error matrix (MSEM) in (1) can be defined as follows:

$$\hat{\alpha}_{OLS} = \Lambda^{-1} X'y \tag{4}$$

$$MSEM\left(\hat{\alpha}_{OLS}\right) = \sigma^2 \Lambda^{-1} \tag{5}$$

For model (3), the following representations can be obtained for the ordinary ridge regression estimator (RE) proposed by Hoerl and Kennard [2], the initial modified two-parameter estimator by Dorugade [11], the Liu estimator [7], Kibria and Lukman estimator [8] respectively:

$$\hat{\alpha}_{RE}(k) = (\lambda + kI)^{-1} Z'y \tag{6}$$

$$MSEM\left(\hat{\alpha}_{RE}(k)\right) = \sigma^2 C_k \Lambda C_k' + k^2 C_k \alpha\alpha' C_k' \tag{7}$$

where $C_k = (\Lambda + kI)^{-1}$

$$\hat{\alpha}_D^*(k,d) = (\Lambda + kdI)^{-1} Z'y \tag{8}$$

$$MSEM\left[\hat{\alpha}_D(k,d)\right] = \sigma^2 R_k \Lambda^{-1} R_k' + (R_k - 1)\alpha\alpha'(R_k - 1)' \tag{9}$$

where $R_k = \Lambda(\Lambda + kdI)^{-1}$

$$\hat{\alpha}_{LE}(d) = [\Lambda + I]^{-1}[\Lambda + dI]\hat{\alpha}_{OLS} \tag{10}$$

$$MSEM\left(\hat{\alpha}_{LE}(d)\right) = \sigma^2 A_d \Lambda^{-1} A_d + (1-d)^2 (\Lambda + I)^{-1} \alpha\alpha' (\Lambda + I)^{-1} \tag{11}$$

where $A_d = (\Lambda + 1)^{-1}(\Lambda + dI)$

$$\hat{\alpha}_{KL}(k) = (\Lambda + kI)^{-1}(\Lambda - kI)\hat{\alpha}_{OLS} \tag{12}$$

$$MSEM\left[\hat{\alpha}_{KL}(k,d)\right] = \sigma^2 W(k) M(k) S^{-1} W'(k) M'(k) + [W(k)M(k) - I]\alpha\alpha'[W(k)M(k) - I]' \tag{13}$$

where $W(k) = (S + kI)^{-1}$, $M(k) = (S + kI)$

2.2. The Proposed Modified Two-parameter Estimator (MTPE) with Prior Information

Swindel [5] proposed a modified ridge estimator based on prior information b as follows:

$$\hat{\alpha}_{MRE}(k,b) = (\Lambda + kI)^{-1}(Z'y + kb) \tag{14}$$

Following Owolabi et al. [17] and Swindel [5], the proposed estimator with prior information is as follows:

$$\hat{\alpha}_{MTPE}(k,d,b) = (\Lambda + (k+d)I)^{-1}(Z'y + (k+d)b) \tag{15}$$

where b = $Q^T b_0$

The following cases hold:

CASE 1: $\hat{\alpha}_{MTPE}(0,0,b) = \hat{\alpha}$

CASE 2: $\hat{\alpha}_{MTPE}(k,d,0) = \hat{\alpha}_{p1}(k,d)$

CASE 3: $\hat{\alpha}_{MTPE}(k,0,0) = \hat{\alpha}_{RE}(k)$

Consequently, the proposed modified two-parameter is a general estimator that includes OLS, the Ridge estimator, and the two-parameter estimator proposed by Owolabi *et al.* [17], as seen in cases 1, 2, and 3, respectively.

The expectation, bias vector, and the covariance matrix of$\alpha_{MTPE}(k,d,b)$are:

$$E\left(\alpha_{MTPE}(k,d,b)\right) = E(\Lambda + (k+d)I)^{-1}(Z'y + (k+d)b)$$

$$= (\Lambda + (k+d)I)^{-1}[\Lambda\alpha + (k+d)b] \tag{16}$$

$$Bias\left(\alpha_{MTPE}(k,d,b)\right) = E\left(\alpha_{MTPE}(k,d,b)\right) - \alpha \tag{17}$$

$$= (\Lambda + (k+d)I)^{-1}[\Lambda\alpha + (k+d)b] - \alpha$$

$$= -(k+d)R_k[\alpha - b] \tag{18}$$

Then, (18) becomes

$$= R_k(k+d)(b-\alpha) = -R_k(k+d)(\alpha - b) \tag{19}$$

The covariance is

$$V\left(\alpha_{MTPE}(k,d,b)\right) = V(\Lambda + (k+d)I)^{-1}(Z'y + (k+d)b)$$

$$= R_k v(y) Z'ZR_k'$$

$$= R_k \sigma^2 Z'ZR_k'$$

$$= R_k \sigma^2 \Lambda R_k' \tag{20}$$

Consequently

$$MSEM\left(\alpha_{MTPE}(k,d,b)\right) = \sigma^2 R_k \Lambda R_k' + R_k(k+d)^2(\alpha - b)(\alpha - b)' R_k' \tag{21}$$

For the convenience of proving the statistical property of the newly proposed Modified Two-Parameter estimator $\hat{\alpha}_{MTPE}(k, d)$, a few important definitions and lemmas become useful.

Lemma 1: Let M be a positive definite matrix, namely M >0, and let αbe some vector, then M - $\alpha\alpha'\geq$ 0if and only if $\alpha' M^{-1}\alpha \leq 1$ [20].

Lemma 2: Let $\hat{\beta}_j = A_j y$, $j = 1, 2$ be two competing estimators of β. Suppose that D = Cov($\hat{\beta}_1$)- Cov($\hat{\beta}_2$)> 0, where Cov($\hat{\beta}_j$), j = 1,2 denotes the covariance matrix of $\hat{\beta}_j$. Then $\Delta \left(\hat{\beta}_1, \hat{\beta}_2\right) = MSEM \left(\hat{\beta}_1\right) - MSEM \left(\hat{\beta}_2\right) \geq 0$if and only if $d_2' \left(D + d_1 d_1'\right)^{-1} d_2 \leq 1$,where MSEM ($\hat{\beta}_j$) and d_j denote the mean squared error matrix and bias vector of $\hat{\beta}_j$, respectively [21].

2.3. Theoretical Comparison

Using Lemma 1 and 2, the theoretical superiority of $\hat{\alpha}_{MTPE}(k, d, b)$over other existing estimators is derived.

2.3.1. Comparison between $\hat{\alpha}_{MTPE}(k, d, b)$ and $\hat{\alpha}_{OLS}$

The MSEM difference between (5) and (21)

$$MSEM(\hat{\alpha}) - MSEM\ (\hat{\alpha}_{MTPE}(k, d, b))$$

$$= \sigma^2\left(\Lambda^{-1} - R_k \Lambda R_k'\right) - R_k (k + d)^2 (\alpha - b)(\alpha - b)' R_k'$$

Now, let k > 0 and 0 < d < 1. Thus, the following theorem holds.

Theorem 1: $MSEM(\hat{\alpha}) - MSEM\ (\hat{\alpha}_{MTPE}(k, d, b))> 0$ if and only if

$$(\alpha - b)' R_k' (k + d)\left[\sigma^2\left(\Lambda^{-1} - R_k \Lambda R_k'\right)\right]^{-1} (k + d) R_k (\alpha - b) \leq 1$$

Proof

$$Cov(\hat{\alpha}) - Cov\ (\hat{\alpha}_{MTPE}(k, d, b))$$

$$= \sigma^2\left(\Lambda^{-1} - R_k \Lambda R_k'\right)$$

$$= \sigma^2 diag\left(\frac{1}{\lambda_i} - \frac{\lambda_i}{(\lambda_i + k + d)^2}\right)$$

It is observed that $\Lambda^{-1} - R_k \Lambda R_k'$ will be positive definite if and only if

$$(\lambda_i + k + d)^2 - \lambda_i{}^2 > 0$$

For 0 < d < 1 and k > 0, $(\lambda_i + k + d)^2 > \lambda_i$. Therefore, $\Lambda^{-1} - R_k \Lambda R_k'$is positive definite. By lemma 3.2, the proof is completed.

2.3.2. Comparison between $\hat{\alpha}_{MTPE}(k, d, b)$ and $\hat{\alpha}_{RE}(k)$

Theorem 2: $\hat{\alpha}_{MTPE}(k, d, b)$ is superior to the ridge estimator $\hat{\alpha}_{RE}(k)$in the MSEM sense if and only if

$$(\alpha - b)' R_k' (k + d)\left[\sigma^2 D_1 + b_1 b_1'\right]^{-1} (k + d) R_k (\alpha - b) \leq 1$$

Proof

$$= V(\hat{\alpha}_{RE}(k)) - V(\hat{\alpha}_{MTPE}(k, d, b))$$

$$= \sigma^2 diag\left[\frac{\lambda_i}{(\lambda_i + k)^2} - \frac{\lambda_i}{(\lambda_i + k + d)^2}\right]$$

Hence, $= V(\hat{\alpha}_{RE}(k)) - V(\hat{\alpha}_{MTPE}(k, d, b))$will be positive definite if and only if

$\lambda_i (\lambda_i + k + d)^2 - \lambda_i (\lambda_i + k)^2 > 0$. By lemma 3.2, the proof is completed.

2.3.3. Comparison between $\hat{\alpha}_{MTPE}(k,d,b)$ and $\hat{\alpha}_{LE}(d)$

Theorem 3: $\hat{\alpha}_{MTPE}(k,d,b)$is superior to the Liu estimator $\hat{\alpha}_{LE}(d)$in the MSEM sense if and only if

$$(\alpha-b)' R_k'(k+d)\left[\sigma^2 D + b_1 b_1'\right]^{-1}(k+d)R_k(\alpha-b) \le 1$$

where $D = A_d\Lambda^{-1}A_d - R_kR_k'$

$$b_1b_1' = (A_d - I)\alpha\alpha'(A_d-I)'$$

Proof

$$V(\hat{\alpha}_{LE}(d)) - V(\hat{\alpha}_{MTPE}(k,d,b))$$
$$= \sigma^2\left(A_d\Lambda^{-1}A_d - R_k\Lambda R_k'\right)$$
$$= \sigma^2 diag\left[\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2} - \frac{\lambda_i}{(\lambda_i+k+d)^2}\right]$$

$V(\hat{\alpha}_{LE}(d)) - V\left(\hat{\alpha}_{mp1}(k,d,b)\right)$ will be positive definite if and only if $(\lambda_i+d)^2(\lambda_i+k+d)^2 - \lambda_i^2(\lambda_i+1)^2 > 0$. By lemma 3.2, the proof is completed.

2.3.4. Comparison between $\hat{\alpha}_{MTPE}(k,d,b)$ and $\hat{\alpha}_D(k,d)$

Theorem 4: $\hat{\alpha}_{MTPE}(k,d,b)$is superior to $\hat{\alpha}_D(k,d)$in the MSEM sense if and only if

$$(\alpha-b)' R_k'(k+d)\left[\sigma^2 D + \left(\tilde{R}_k\Lambda - I\right)\alpha\alpha'\left(\tilde{R}_k\Lambda - I\right)\right]^{-1}(k+d)R_k(\alpha-b) \le 1$$

where $D = \tilde{R}_k\tilde{R}_k' - R_kR_k'$

Proof

$$V(\hat{\alpha}_D(k,d)) - V(\hat{\alpha}_{MTPE}(k,d,b))$$
$$= \tilde{R}_k\Lambda\tilde{R}_k' - R_k\Lambda R_k'$$
$$= \sigma^2 diag\left[\frac{\lambda_i}{(\lambda_i+kd)^2} - \frac{\lambda_i}{(\lambda_i+k+d)^2}\right]$$

$V(\hat{\alpha}_D(k,d)) - V(\hat{\alpha}_{MTPE}(k,d,b) > 0)$if and only if $(\lambda_i+k+d)^2 - (\lambda_i+kd)^2 > 0$. By lemma 3.2, the proof is completed.

2.3.5. Comparison between $\hat{\alpha}_{MTPE}(k,d,b)$ and $\hat{\alpha}_{KL}$

Theorem: $\hat{\alpha}_{MTPE}(k,d,b)$ is superior to $\hat{\alpha}_{KL}$in the MSEM sense if and only if

$$(\alpha-b)' R_k'(k+d)\left[\sigma^2\left(W(k)M(k)S^{-1}W'(k)M'(k) - R_k\Lambda R_k'\right) + (W(k)M(k)-I)\alpha\alpha'(W(k)M(k)-I)'\right]^{-1}$$
$$(\alpha-b)R_k(k+d)$$

Proof

$$V(\hat{\alpha}_{KL}) - V(\hat{\alpha}_{MTPE}(k,d,b))$$
$$= \sigma^2 diag\left[\frac{(\lambda_i-k)^2}{\lambda_i(\lambda_i+k)^2} - \frac{\lambda_i}{(\lambda_i+k+d)^2}\right]$$

$V(\hat{\alpha}_{KL}) - V(\hat{\alpha}_{MTPE}(k,d,b)) > 0$if and only if $(\lambda_i+k+d)^2(\lambda_i-k)^2 - \lambda_i^2(\lambda_i+k)^2 > 0$. By lemma 3.2, the proof is completed.

Obviously, $\Delta_1 \ge 0$if and only if $\alpha\alpha' - (\alpha-b)(\alpha-b)' \ge 0$, thus, the following results hold.

2.4. Computation of parameters k and d

The selection of the biasing ridge parameter *k is* crucial in this study because it controls the bias of the regression towards the mean of the response variable. Several methods for estimating *k* have been proposed. They include Hoerl and Kennard [2], Kibria [22], Lukman *et al.* [23], Saleh *et al.* (2019), Dawoud and Kibria [24] and some others. In the same vein, *d* is another biasing parameter whenever we have two-parameter estimators (Liu, [7]). Other authors who have discussed the biasing parameter of the Liu estimator are Akdeniz and Kaciranlar [25], Kaciranlar [26], Alheety and Kibria [27], Shukur *et al.* [28] and Qasim, Amin, and Ullah [29], Månsson *et al.* [30]. This study adopts the biasing parameters *k* and *d* used by Owolabi *et al.* [17]

$$\hat{k} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d \tag{22}$$

Different forms of biasing parameter k in equation (22) will be used in this work as adopted by Lukman and Ayinde [31] and Fayose and Ayinde [19]. These forms are measures of location like Mid-Range (MR), Arithmetic Mean (AM), Median (MED), Geometric Mean (GM), Harmonic Mean (HM), and then the generalized form.

$$\hat{k}_{MR} = \frac{Max\left[\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d\right] - Min\left[\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d\right]}{2} \tag{23}$$

$$\hat{k}_{MED} = Median\left[\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d\right] \tag{24}$$

$$\hat{k}_{AM} = \frac{1}{p}\sum_{i=1}^{p}\left[\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d\right] \tag{25}$$

$$\hat{k}_{GM} = \left[\prod_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - d\right]^{1/p} \tag{26}$$

$$\hat{k}_{HM} = \left[\frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2} - d\right] \tag{27}$$

The biasing parameter from Owolabi *et al.* [17] is:

$$\hat{d} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - k \tag{28}$$

$$\hat{d}^* = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} - k\right) \tag{29}$$

In case $\hat{d}^*$ is not between 0 and 1, use $\hat{d}^* = 0$

2.5. Simulation Study

In this section, a Monte Carlo simulation study is performed to show the performance of the proposed estimator with prior information over some existing estimators. The explanatory variables were generated using the method used by McDonald and Galarneau [32], Wichern and Churchill [33], Gibbons [34], and Lukman *et al.* [35] were used to generate explanatory variables in this study: This is given as:

$$x_{ij} = \left(1 - \rho^2\right)^{1/2} z_{ij} + \rho z_{i,p+1} i = 1, 2, ..., n, j = 1, 2, ..., p. \tag{30}$$

where ρ is the correlation between any two independent variables, p is the number of the independent variable, and z_{ij} is an independent standard normal distribution with mean zero and variance one. The values of ρ consider in this study are 0.8, 0.95, and 0.999, while the sample sizes are 20, 50, and 100. Also, explanatory variables (p) will be taken to be three (3) and seven (7) for the simulation study. The error terms,u_t, will be generated following Firinguetti [36] such that $u_t \sim N\left(0, \sigma^2 I\right)$. The values of $\beta'\beta = 1$ [37]. The standard deviations in this simulation study are σ = 1, 5, and 10. The replication for the study is 1000 times. The mean square error was obtained using the following:

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\alpha}_{ij} - \alpha_i\right)'\left(\hat{\alpha}_{ij} - \alpha_i\right) \tag{31}$$

3. Results from Simulation Study and Discussions

The results obtained from the simulation study at the different specifications of error variance, multicollinearity levels, sample sizes, and numbers of regressors are presented in section 3.1, while the discussions of the simulation results are presented in 3.2.

3.1. Results from Simulation Study

Tables 1 to 6 provide the summary of the results as follows:

Table 1: Estimated MSE for MTPE with arithmetic mean version of k and other estimators

P	N	σ	ρ	OLS	LIU	RE	DE	KL	NRTE	MTPAM
3	20	1	0.8	3.4392	3.3502	3.1536	3.3351	2.8863	3.2064	**2.5068**
			0.95	4.2482	3.8164	3.5004	3.8659	3.8707	3.6073	**3.1259**
			0.999	55.1758	**3.3769**	18.3562	50.833	55.1719	27.3905	11.0196
		3	0.8	5.9963	5.2767	3.7119	5.8671	**2.7006**	4.0919	3.2715
			0.95	13.0283	9.0355	6.0092	10.0196	10.5066	7.4106	**5.0269**
			0.999	471.1307	**5.6311**	138.515	431.3514	471.1131	221.1737	57.9329
		10	0.8	35.0959	27.7936	12.8235	33.6529	14.758	17.2883	**9.4087**
			0.95	112.8854	69.2027	35.8961	78.5946	97.6977	53.6148	**20.0765**
			0.999	5202.07	**31.3641**	1504.267	4759.287	5202.013	2427.795	589.0742
	50	1	0.8	3.273	3.2476	3.1567	3.2302	2.7749	3.1798	**2.3987**
			0.95	3.5476	3.4656	3.3089	3.4473	3.1822	3.3097	**2.8634**
			0.999	21.9106	**3.6598**	8.4126	19.0032	21.9119	11.626	6.7705
		3	0.8	4.1581	4.0239	3.1779	4.1581	**1.9745**	3.3645	2.7107
			0.95	6.6589	5.701	4.0668	6.3989	**3.5504**	4.5355	3.7736
			0.999	172.5613	**7.871**	49.8762	145.5923	172.539	79.9239	24.2571
		10	0.8	14.6254	13.4712	6.226	14.6254	**4.5343**	7.7328	5.2577
			0.95	42.5129	32.134	14.2641	39.4158	21.7407	20.4137	**9.7863**
			0.999	1887.896	**56.3318**	522.9142	1586.947	1887.785	859.5952	216.9213
	100	1	0.8	3.356	3.3464	3.3067	3.336	2.9776	3.3045	**2.5726**
			0.95	3.4485	3.4187	3.3413	3.4035	3.1971	3.3097	**2.9158**
			0.999	13.8645	**3.979**	6.4385	11.6463	13.8399	8.1595	5.7382
		3	0.8	3.8814	3.8356	3.3639	3.8814	**2.2918**	3.4572	2.7873
			0.95	5.2518	4.8797	3.8434	5.229	**2.9822**	4.0778	3.5576
			0.999	98.6581	**10.0305**	31.3298	78.1771	98.4632	47.8308	16.129
		10	0.8	9.7129	9.3557	4.9706	9.7129	**3.0605**	5.7565	4.3562
			0.95	25.4599	21.482	9.8547	25.2036	9.1384	13.1423	**7.2828**
			0.999	1061.012	**78.4863**	314.0488	832.9202	1060.216	498.6753	124.4761
7	20	1	0.8	4.239	3.806	3.1913	3.7132	3.7466	3.4427	**1.9558**
			0.95	8.2215	4.8562	4.3135	6.2348	8.0833	5.3717	**2.6403**
			0.999	271.2359	**3.0537**	73.3716	262.8589	271.2359	132.4997	13.4471
		3	0.8	16.2106	11.8556	6.3453	12.5721	12.1663	9.0525	**2.8205**

			0.95	52.2055	21.5665	15.9005	33.7856	50.8362	26.4852	**5.2482**
			0.999	2421.699	**5.698**	638.3646	2345.926	2421.699	1172.073	61.176
		10	0.8	152.7874	104.4895	43.1458	111.9801	119.0785	74.8448	**9.5574**
			0.95	553.3428	212.5212	148.3353	346.9086	541.659	268.4885	**22.0999**
			0.999	26890.13	**36.0226**	7068.979	26047.81	26890.13	13002.6	570.0013
	50	1	0.8	3.0127	2.9799	2.7991	2.9294	2.5421	2.7977	**1.8483**
			0.95	4.0714	3.8165	3.2156	3.6192	3.6355	3.3393	**2.3573**
			0.999	69.0829	**3.6436**	20.3854	63.3417	69.0834	33.6802	6.9469
		3	0.8	6.0401	5.6186	3.5748	6.0356	2.5295	4.1444	**2.1534**
			0.95	14.9977	11.812	6.1234	12.5007	9.4671	8.3812	**3.3351**
			0.999	599.3054	**10.7486**	159.4636	546.7704	599.3068	280.4733	21.5339
		10	0.8	40.2032	35.644	12.7445	40.1489	12.7329	19.7139	**4.6031**
			0.95	139.2015	103.8733	38.9681	110.8799	88.6376	65.859	**9.3069**
			0.999	6629.406	**91.2743**	1740.234	6044.489	6629.408	3086.268	148.9701
	100	1	0.8	2.9321	2.9219	2.8429	2.8961	2.5434	2.8212	**1.8514**
			0.95	3.4372	3.3549	3.0527	3.2601	3.0723	3.0665	**2.2903**
			0.999	37.6542	**4.1778**	12.3	32.9267	37.653	19.264	5.1735
		3	0.8	4.5006	4.3811	3.2633	4.5006	2.327	3.5231	**1.998**
			0.95	9.1339	7.8919	4.5997	8.8268	4.4469	5.7377	**2.7772**
			0.999	317.3564	15.6873	87.4449	273.7222	317.3513	151.9645	**14.3249**
		10	0.8	22.3369	21.0644	8.1071	22.3369	7.1352	11.697	**3.2938**
			0.95	74.3299	60.5298	22.1861	70.7494	30.4026	36.442	**6.2735**
			0.999	3500.284	146.8468	943.1045	3014.12	3500.256	1663.128	**82.6421**

Table 2: Estimated MSE for MTPE with a generalized version of k and other estimators

P	N	σ	ρ	OLS	LIU	RE	DE	KL	NRTE	MTPGEN
3	20	1	0.8	3.4392	3.3502	3.1536	3.3351	**2.8863**	3.2064	3.2734
			0.95	4.2482	3.8164	**3.5004**	3.8659	3.8707	3.6073	3.7828
			0.999	55.1758	**3.3769**	18.3562	50.833	55.1719	27.3905	34.8212
		3	0.8	5.9963	5.2767	3.7119	5.8671	**2.7006**	4.0919	4.6251
			0.95	13.0283	9.0355	**6.0092**	10.0196	10.5066	7.4106	8.9354
			0.999	471.1307	**5.6311**	138.515	431.3514	471.1131	221.1737	288.008
		10	0.8	35.0959	27.7936	**12.8235**	33.6529	14.758	17.2883	22.0862
			0.95	112.8854	69.2027	**35.8961**	78.5946	97.6977	53.6148	69.4969
			0.999	5202.07	**31.3641**	1504.267	4759.287	5202.013	2427.795	3169.379
	50	1	0.8	3.273	3.2476	3.1567	3.2302	**2.7749**	3.1798	3.2071
			0.95	3.5476	3.4656	3.3089	3.4473	**3.1822**	3.3097	3.375
			0.999	21.9106	**3.6598**	8.4126	19.0032	21.9119	11.626	14.3731
		3	0.8	4.1581	4.0239	3.1779	4.1581	**1.9745**	3.3645	3.5927
			0.95	6.6589	5.701	4.0668	6.3989	**3.5504**	4.5355	5.116
			0.999	172.5613	**7.871**	49.8762	145.5923	172.539	79.9239	104.6721
		10	0.8	14.6254	13.4712	6.226	14.6254	**4.5343**	7.7328	9.6102
			0.95	42.5129	32.134	**14.2641**	39.4158	21.7407	20.4137	26.3502
			0.999	1887.896	**56.3318**	522.9142	1586.947	1887.785	859.5952	1134.225
	100	1	0.8	3.356	3.3464	3.3067	3.336	**2.9776**	3.3045	3.3196
			0.95	3.4485	3.4187	3.3413	3.4035	**3.1971**	3.3097	3.3484
			0.999	13.8645	**3.979**	6.4385	11.6463	13.8399	8.1595	9.6936
		3	0.8	3.8814	3.8356	3.3639	3.8814	**2.2918**	3.4572	3.5797
			0.95	5.2518	4.8797	3.8434	5.229	**2.9822**	4.0778	4.4008

			0.999	98.6581	**10.0305**	31.3298	78.1771	98.4632	47.8308	61.4655
		10	0.8	9.7129	9.3557	4.9706	9.7129	**3.0605**	5.7565	6.8532
			0.95	25.4599	21.482	9.8547	25.2036	**9.1384**	13.1423	16.479
			0.999	1061.012	**78.4863**	314.0488	832.9202	1060.216	498.6753	649.3586
7	20	1	0.8	4.239	3.806	**3.1913**	3.7132	3.7466	3.4427	3.5288
			0.95	8.2215	4.8562	**4.3135**	6.2348	8.0833	5.3717	5.6766
			0.999	271.2359	**3.0537**	73.3716	262.8589	271.2359	132.4997	147.3248
		3	0.8	16.2106	11.8556	**6.3453**	12.5721	12.1663	9.0525	9.8241
			0.95	52.2055	21.5665	**15.9005**	33.7856	50.8362	26.4852	29.2382
			0.999	2421.699	**5.698**	638.3646	2345.926	2421.699	1172.073	1305.642
		10	0.8	152.7874	104.4895	**43.1458**	111.9801	119.0785	74.8448	83.1835
			0.95	553.3428	212.5212	**148.3353**	346.9086	541.659	268.4885	298.9364
			0.999	26890.13	**36.0226**	7068.979	26047.81	26890.13	13002.6	14487.08
	50	1	0.8	3.0127	2.9799	2.7991	2.9294	**2.5421**	2.7977	2.8212
			0.95	4.0714	3.8165	**3.2156**	3.6192	3.6355	3.3393	3.4173
			0.999	69.0829	**3.6436**	20.3854	63.3417	69.0834	33.6802	37.4501
		3	0.8	6.0401	5.6186	3.5748	6.0356	**2.5295**	4.1444	4.35
			0.95	14.9977	11.812	**6.1234**	12.5007	9.4671	8.3812	9.0875
			0.999	599.3054	**10.7486**	159.4636	546.7704	599.3068	280.4733	314.4318
		10	0.8	40.2032	35.644	12.7445	40.1489	**12.7329**	19.7139	21.9094
			0.95	139.2015	103.8733	**38.9681**	110.8799	88.6376	65.859	73.6765
			0.999	6629.406	**91.2743**	1740.234	6044.489	6629.408	3086.268	3463.66
	100	1	0.8	2.9321	2.9219	2.8429	2.8961	**2.5434**	2.8212	2.8334
			0.95	3.4372	3.3549	**3.0527**	3.2601	3.0723	3.0665	3.106
			0.999	37.6542	**4.1778**	12.3	32.9267	37.653	19.264	21.2334
		3	0.8	4.5006	4.3811	3.2633	4.5006	**2.327**	3.5231	3.6299
			0.95	9.1339	7.8919	4.5997	8.8268	**4.4469**	5.7377	6.1015
			0.999	317.3564	**15.6873**	87.4449	273.7222	317.3513	151.9645	169.6724
		10	0.8	22.3369	21.0644	8.1071	22.3369	**7.1352**	11.697	12.8485
			0.95	74.3299	60.5298	**22.1861**	70.7494	30.4026	36.442	40.5045
			0.999	3500.284	**146.8468**	943.1045	3014.12	3500.256	1663.128	1859.804

Table 3: Estimated MSE for MTPE with the harmonic mean version of k and other estimators

P	N	σ	ρ	OLS	LIU	RE	DE	KL	NRTE	MTPHM
3	20	1	0.8	3.4392	3.3502	3.1536	3.3351	**2.8863**	3.2064	3.2901
			0.95	4.2482	3.8164	**3.5004**	3.8659	3.8707	3.6073	3.8355
			0.999	55.1758	**3.3769**	18.3562	50.833	55.1719	27.3905	36.4299
		3	0.8	5.9963	5.2767	3.7119	5.8671	**2.7006**	4.0919	4.391
			0.95	13.0283	9.0355	**6.0092**	10.0196	10.5066	7.4106	9.0261
			0.999	471.1307	**5.6311**	138.515	431.3514	471.1131	221.1737	302.3617
		10	0.8	35.0959	27.7936	**12.8235**	33.6529	14.758	17.2883	19.1008
			0.95	112.8854	69.2027	**35.8961**	78.5946	97.6977	53.6148	69.2965
			0.999	5202.07	**31.3641**	1504.267	4759.287	5202.013	2427.795	3326.587
	50	1	0.8	3.273	3.2476	3.1567	3.2302	**2.7749**	3.1798	3.2165
			0.95	3.5476	3.4656	3.3089	3.4473	**3.1822**	3.3097	3.4174
			0.999	21.9106	**3.6598**	8.4126	19.0032	21.9119	11.626	14.9695
		3	0.8	4.1581	4.0239	3.1779	4.1581	**1.9745**	3.3645	3.467
			0.95	6.6589	5.701	4.0668	6.3989	**3.5504**	4.5355	4.8546
			0.999	172.5613	**7.871**	49.8762	145.5923	172.539	79.9239	110.0578

		10	0.8	14.6254	13.4712	6.226	14.6254	**4.5343**	7.7328	8.407
			0.95	42.5129	32.134	**14.2641**	39.4158	21.7407	20.4137	22.6127
			0.999	1887.896	**56.3318**	522.9142	1586.947	1887.785	859.5952	1193.281
	100	1	0.8	3.356	3.3464	3.3067	3.336	**2.9776**	3.3045	3.333
			0.95	3.4485	3.4187	3.3413	3.4035	**3.1971**	3.3097	3.3934
			0.999	13.8645	**3.979**	6.4385	11.6463	13.8399	8.1595	10.0312
		3	0.8	3.8814	3.8356	3.3639	3.8814	**2.2918**	3.4572	3.5237
			0.95	5.2518	4.8797	3.8434	5.229	**2.9822**	4.0778	4.2414
			0.999	98.6581	**10.0305**	31.3298	78.1771	98.4632	47.8308	64.4895
		10	0.8	9.7129	9.3557	4.9706	9.7129	**3.0605**	5.7565	6.2362
			0.95	25.4599	21.482	9.8547	25.2036	**9.1384**	13.1423	13.9288
			0.999	1061.012	**78.4863**	314.0488	832.9202	1060.216	498.6753	682.751
7	20	1	0.8	4.239	3.806	**3.1913**	3.7132	3.7466	3.4427	3.6437
			0.95	8.2215	4.8562	**4.3135**	6.2348	8.0833	5.3717	6.1865
			0.999	271.2359	**3.0537**	73.3716	262.8589	271.2359	132.4997	172.5299
		3	0.8	16.2106	11.8556	**6.3453**	12.5721	12.1663	9.0525	9.7872
			0.95	52.2055	21.5665	**15.9005**	33.7856	50.8362	26.4852	33.5736
			0.999	2421.699	**5.698**	638.3646	2345.926	2421.699	1172.073	1532.816
		10	0.8	152.7874	104.4895	**43.1458**	111.9801	119.0785	74.8448	81.5896
			0.95	553.3428	212.5212	**148.3353**	346.9086	541.659	268.4885	346.2006
			0.999	26890.13	**36.0226**	7068.979	26047.81	26890.13	13002.6	17011.6
	50	1	0.8	3.0127	2.9799	2.7991	2.9294	**2.5421**	2.7977	2.8898
			0.95	4.0714	3.8165	**3.2156**	3.6192	3.6355	3.3393	3.5607
			0.999	69.0829	**3.6436**	20.3854	63.3417	69.0834	33.6802	44.5708
		3	0.8	6.0401	5.6186	3.5748	6.0356	**2.5295**	4.1444	3.8419
			0.95	14.9977	11.812	**6.1234**	12.5007	9.4671	8.3812	8.5668
			0.999	599.3054	**10.7486**	159.4636	546.7704	599.3068	280.4733	378.6472
		10	0.8	40.2032	35.644	12.7445	40.1489	**12.7329**	19.7139	15.4915
			0.95	139.2015	103.8733	**38.9681**	110.8799	88.6376	65.859	66.6129
			0.999	6629.406	**91.2743**	1740.234	6044.489	6629.408	3086.268	4177.451
	100	1	0.8	2.9321	2.9219	2.8429	2.8961	**2.5434**	2.8212	2.883
			0.95	3.4372	3.3549	**3.0527**	3.2601	3.0723	3.0665	3.2073
			0.999	37.6542	**4.1778**	12.3	32.9267	37.653	19.264	25.1235
		3	0.8	4.5006	4.3811	3.2633	4.5006	**2.327**	3.5231	3.4035
			0.95	9.1339	7.8919	4.5997	8.8268	**4.4469**	5.7377	5.2723
			0.999	317.3564	**15.6873**	87.4449	273.7222	317.3513	151.9645	204.5296
		10	0.8	22.3369	21.0644	8.1071	22.3369	**7.1352**	11.697	9.5356
			0.95	74.3299	60.5298	**22.1861**	70.7494	30.4026	36.442	29.7829
			0.999	3500.284	**146.8468**	943.1045	3014.12	3500.256	1663.128	2246.449

Table 4: Estimated MSE for MTPE with the geometric mean version of k and other estimators

P	N	σ	ρ	OLS	LIU	RE	DE	KL	NRTE	MTPGM
3	20	1	0.8	3.4392	3.3502	3.1536	3.3351	**2.8863**	3.2064	3.1506
			0.95	4.2482	3.8164	**3.5004**	3.8659	3.8707	3.6073	3.7717
			0.999	55.1758	**3.3769**	18.3562	50.833	55.1719	27.3905	36.429
		3	0.8	5.9963	5.2767	3.7119	5.8671	**2.7006**	4.0919	3.8836
			0.95	13.0283	9.0355	**6.0092**	10.0196	10.5066	7.4106	8.7321
			0.999	471.1307	**5.6311**	138.515	431.3514	471.1131	221.1737	302.3597
		10	0.8	35.0959	27.7936	**12.8235**	33.6529	14.758	17.2883	14.3915

			0.95	112.8854	69.2027	**35.8961**	78.5946	97.6977	53.6148	65.8791
			0.999	5202.07	**31.3641**	1504.267	4759.287	5202.013	2427.795	3326.575
	50	1	0.8	3.273	3.2476	3.1567	3.2302	**2.7749**	3.1798	3.058
			0.95	3.5476	3.4656	3.3089	3.4473	**3.1822**	3.3097	3.3315
			0.999	21.9106	**3.6598**	8.4126	19.0032	21.9119	11.626	14.9674
		3	0.8	4.1581	4.0239	3.1779	4.1581	**1.9745**	3.3645	3.0844
			0.95	6.6589	5.701	4.0668	6.3989	**3.5504**	4.5355	4.4593
			0.999	172.5613	**7.871**	49.8762	145.5923	172.539	79.9239	110.0517
		10	0.8	14.6254	13.4712	6.226	14.6254	**4.5343**	7.7328	6.3798
			0.95	42.5129	32.134	**14.2641**	39.4158	21.7407	20.4137	17.041
			0.999	1887.896	**56.3318**	522.9142	1586.947	1887.785	859.5952	1193.239
	100	1	0.8	3.356	3.3464	3.3067	3.336	**2.9776**	3.3045	3.2056
			0.95	3.4485	3.4187	3.3413	3.4035	**3.1971**	3.3097	3.3269
			0.999	13.8645	**3.979**	6.4385	11.6463	13.8399	8.1595	10.0293
		3	0.8	3.8814	3.8356	3.3639	3.8814	**2.2918**	3.4572	3.1822
			0.95	5.2518	4.8797	3.8434	5.229	**2.9822**	4.0778	3.9439
			0.999	98.6581	**10.0305**	31.3298	78.1771	98.4632	47.8308	64.4796
		10	0.8	9.7129	9.3557	4.9706	9.7129	**3.0605**	5.7565	5.1005
			0.95	25.4599	21.482	9.8547	25.2036	**9.1384**	13.1423	9.9694
			0.999	1061.012	**78.4863**	314.0488	832.9202	1060.216	498.6753	682.6299
7	20	1	0.8	4.239	3.806	**3.1913**	3.7132	3.7466	3.4427	3.4906
			0.95	8.2215	4.8562	**4.3135**	6.2348	8.0833	5.3717	6.1589
			0.999	271.2359	**3.0537**	73.3716	262.8589	271.2359	132.4997	172.5299
		3	0.8	16.2106	11.8556	**6.3453**	12.5721	12.1663	9.0525	8.8997
			0.95	52.2055	21.5665	**15.9005**	33.7856	50.8362	26.4852	33.3637
			0.999	2421.699	**5.698**	638.3646	2345.926	2421.699	1172.073	1532.816
		10	0.8	152.7874	104.4895	**43.1458**	111.9801	119.0785	74.8448	72.2813
			0.95	553.3428	212.5212	**148.3353**	346.9086	541.659	268.4885	343.6968
			0.999	26890.13	**36.0226**	7068.979	26047.81	26890.13	13002.6	17011.6
	50	1	0.8	3.0127	2.9799	2.7991	2.9294	**2.5421**	2.7977	2.6999
			0.95	4.0714	3.8165	**3.2156**	3.6192	3.6355	3.3393	3.444
			0.999	69.0829	**3.6436**	20.3854	63.3417	69.0834	33.6802	44.5707
		3	0.8	6.0401	5.6186	3.5748	6.0356	**2.5295**	4.1444	3.0026
			0.95	14.9977	11.812	**6.1234**	12.5007	9.4671	8.3812	7.5469
			0.999	599.3054	**10.7486**	159.4636	546.7704	599.3068	280.4733	378.6471
		10	0.8	40.2032	35.644	12.7445	40.1489	12.7329	19.7139	**8.2605**
			0.95	139.2015	103.8733	**38.9681**	110.8799	88.6376	65.859	53.9866
			0.999	6629.406	**91.2743**	1740.234	6044.489	6629.408	3086.268	4177.451
	100	1	0.8	2.9321	2.9219	2.8429	2.8961	**2.5434**	2.8212	2.7132
			0.95	3.4372	3.3549	**3.0527**	3.2601	3.0723	3.0665	3.0838
			0.999	37.6542	**4.1778**	12.3	32.9267	37.653	19.264	25.1235
		3	0.8	4.5006	4.3811	3.2633	4.5006	**2.327**	3.5231	2.7722
			0.95	9.1339	7.8919	4.5997	8.8268	4.4469	5.7377	**4.2117**
			0.999	317.3564	**15.6873**	87.4449	273.7222	317.3513	151.9645	204.5296
		10	0.8	22.3369	21.0644	8.1071	22.3369	7.1352	11.697	**5.5008**
			0.95	74.3299	60.5298	22.1861	70.7494	30.4026	36.442	**17.3761**
			0.999	3500.284	**146.8468**	943.1045	3014.12	3500.256	1663.128	2246.448

Table 5: Estimated MSE for MTPE with the median version of k and other estimators

P	N	σ	ρ	OLS	LIU	RE	DE	KL	NRTE	MTPMD
3	20	1	0.8	3.4392	3.3502	3.1536	3.3351	**2.8863**	3.2064	2.9472
			0.95	4.2482	3.8164	**3.5004**	3.8659	3.8707	3.6073	3.6154
			0.999	55.1758	**3.3769**	18.3562	50.833	55.1719	27.3905	19.9157
		3	0.8	5.9963	5.2767	3.7119	5.8671	**2.7006**	4.0919	3.9242
			0.95	13.0283	9.0355	**6.0092**	10.0196	10.5066	7.4106	6.3497
			0.999	471.1307	**5.6311**	138.515	431.3514	471.1131	221.1737	152.9624
		10	0.8	35.0959	27.7936	**12.8235**	33.6529	14.758	17.2883	13.8106
			0.95	112.8854	69.2027	**35.8961**	78.5946	97.6977	53.6148	38.5191
			0.999	5202.07	**31.3641**	1504.267	4759.287	5202.013	2427.795	1666.585
	50	1	0.8	3.273	3.2476	3.1567	3.2302	**2.7749**	3.1798	2.7838
			0.95	3.5476	3.4656	3.3089	3.4473	**3.1822**	3.3097	3.2347
			0.999	21.9106	**3.6598**	8.4126	19.0032	21.9119	11.626	9.332
		3	0.8	4.1581	4.0239	3.1779	4.1581	**1.9745**	3.3645	3.2347
			0.95	6.6589	5.701	4.0668	6.3989	**3.5504**	4.5355	4.3249
			0.999	172.5613	**7.871**	49.8762	145.5923	172.539	79.9239	57.955
		10	0.8	14.6254	13.4712	6.226	14.6254	**4.5343**	7.7328	6.9136
			0.95	42.5129	32.134	**14.2641**	39.4158	21.7407	20.4137	15.5612
			0.999	1887.896	**56.3318**	522.9142	1586.947	1887.785	859.5952	613.7501
	100	1	0.8	3.356	3.3464	3.3067	3.336	2.9776	3.3045	**2.9448**
			0.95	3.4485	3.4187	3.3413	3.4035	**3.1971**	3.3097	3.2243
			0.999	13.8645	**3.979**	6.4385	11.6463	13.8399	8.1595	6.9603
		3	0.8	3.8814	3.8356	3.3639	3.8814	**2.2918**	3.4572	3.2627
			0.95	5.2518	4.8797	3.8434	5.229	**2.9822**	4.0778	4.0307
			0.999	98.6581	**10.0305**	31.3298	78.1771	98.4632	47.8308	35.0725
		10	0.8	9.7129	9.3557	4.9706	9.7129	**3.0605**	5.7565	5.3898
			0.95	25.4599	21.482	9.8547	25.2036	**9.1384**	13.1423	10.4989
			0.999	1061.012	**78.4863**	314.0488	832.9202	1060.216	498.6753	355.0364
7	20	1	0.8	4.239	3.806	3.1913	3.7132	3.7466	3.4427	**2.8343**
			0.95	8.2215	4.8562	4.3135	6.2348	8.0833	5.3717	**3.9106**
			0.999	271.2359	**3.0537**	73.3716	262.8589	271.2359	132.4997	42.0146
		3	0.8	16.2106	11.8556	6.3453	12.5721	12.1663	9.0525	**4.9461**
			0.95	52.2055	21.5665	15.9005	33.7856	50.8362	26.4852	**10.1726**
			0.999	2421.699	**5.698**	638.3646	2345.926	2421.699	1172.073	357.3332
		10	0.8	152.7874	104.4895	43.1458	111.9801	119.0785	74.8448	**25.51**
			0.95	553.3428	212.5212	148.3353	346.9086	541.659	268.4885	**84.1308**
			0.999	26890.13	**36.0226**	7068.979	26047.81	26890.13	13002.6	3947.902
	50	1	0.8	3.0127	2.9799	2.7991	2.9294	2.5421	2.7977	**2.4677**
			0.95	4.0714	3.8165	3.2156	3.6192	3.6355	3.3393	**3.0447**
			0.999	69.0829	**3.6436**	20.3854	63.3417	69.0834	33.6802	15.0695
		3	0.8	6.0401	5.6186	3.5748	6.0356	**2.5295**	4.1444	3.3346
			0.95	14.9977	11.812	6.1234	12.5007	9.4671	8.3812	**5.268**
			0.999	599.3054	**10.7486**	159.4636	546.7704	599.3068	280.4733	112.6137
		10	0.8	40.2032	35.644	12.7445	40.1489	12.7329	19.7139	**9.681**
			0.95	139.2015	103.8733	38.9681	110.8799	88.6376	65.859	**27.3734**
			0.999	6629.406	**91.2743**	1740.234	6044.489	6629.408	3086.268	1221.81
	100	1	0.8	2.9321	2.9219	2.8429	2.8961	2.5434	2.8212	**2.5205**
			0.95	3.4372	3.3549	3.0527	3.2601	3.0723	3.0665	**2.8763**
			0.999	37.6542	**4.1778**	12.3	32.9267	37.653	19.264	9.2368

		3	0.8	4.5006	4.3811	3.2633	4.5006	**2.327**	3.5231	2.9934
			0.95	9.1339	7.8919	4.5997	8.8268	4.4469	5.7377	**4.2915**
			0.999	317.3564	**15.6873**	87.4449	273.7222	317.3513	151.9645	59.9551
		10	0.8	22.3369	21.0644	8.1071	22.3369	7.1352	11.697	**6.5125**
			0.95	74.3299	60.5298	22.1861	70.7494	30.4026	36.442	**15.6072**
			0.999	3500.284	**146.8468**	943.1045	3014.12	3500.256	1663.128	639.3099

Table 6: Estimated MSE for MTPE with the mid-range version of k and other estimators

P	N	σ	ρ	OLS	LIU	RE	DE	KL	NRTE	MTPMR
3	20	1	0.8	3.4392	3.3502	3.1536	3.3351	2.8863	3.2064	**2.4497**
			0.95	4.2482	3.8164	3.5004	3.8659	3.8707	3.6073	**3.0588**
			0.999	55.1758	**3.3769**	18.3562	50.833	55.1719	27.3905	10.5112
		3	0.8	5.9963	5.2767	3.7119	5.8671	**2.7006**	4.0919	3.1946
			0.95	13.0283	9.0355	6.0092	10.0196	10.5066	7.4106	**4.8678**
			0.999	471.1307	**5.6311**	138.515	431.3514	471.1131	221.1737	57.0671
		10	0.8	35.0959	27.7936	12.8235	33.6529	14.758	17.2883	**9.0448**
			0.95	112.8854	69.2027	35.8961	78.5946	97.6977	53.6148	**19.2164**
			0.999	5202.07	**31.3641**	1504.267	4759.287	5202.013	2427.795	586.6799
	50	1	0.8	3.273	3.2476	3.1567	3.2302	2.7749	3.1798	**2.3524**
			0.95	3.5476	3.4656	3.3089	3.4473	3.1822	3.3097	**2.8073**
			0.999	21.9106	**3.6598**	8.4126	19.0032	21.9119	11.626	6.4893
		3	0.8	4.1581	4.0239	3.1779	4.1581	**1.9745**	3.3645	2.6451
			0.95	6.6589	5.701	4.0668	6.3989	**3.5504**	4.5355	3.6961
			0.999	172.5613	**7.871**	49.8762	145.5923	172.539	79.9239	23.5858
		10	0.8	14.6254	13.4712	6.226	14.6254	**4.5343**	7.7328	5.0884
			0.95	42.5129	32.134	14.2641	39.4158	21.7407	20.4137	**9.3773**
			0.999	1887.896	**56.3318**	522.9142	1586.947	1887.785	859.5952	215.4871
	100	1	0.8	3.356	3.3464	3.3067	3.336	2.9776	3.3045	**2.5227**
			0.95	3.4485	3.4187	3.3413	3.4035	3.1971	3.3097	**2.8662**
			0.999	13.8645	**3.979**	6.4385	11.6463	13.8399	8.1595	5.5538
		3	0.8	3.8814	3.8356	3.3639	3.8814	**2.2918**	3.4572	2.7231
			0.95	5.2518	4.8797	3.8434	5.229	**2.9822**	4.0778	3.4892
			0.999	98.6581	**10.0305**	31.3298	78.1771	98.4632	47.8308	15.516
		10	0.8	9.7129	9.3557	4.9706	9.7129	**3.0605**	5.7565	4.2361
			0.95	25.4599	21.482	9.8547	25.2036	9.1384	13.1423	**6.982**
			0.999	1061.012	**78.4863**	314.0488	832.9202	1060.216	498.6753	123.3963
7	20	1	0.8	4.239	3.806	3.1913	3.7132	3.7466	3.4427	**1.7338**
			0.95	8.2215	4.8562	4.3135	6.2348	8.0833	5.3717	**2.3315**
			0.999	271.2359	**3.0537**	73.3716	262.8589	271.2359	132.4997	10.3911
		3	0.8	16.2106	11.8556	6.3453	12.5721	12.1663	9.0525	**2.3746**
			0.95	52.2055	21.5665	15.9005	33.7856	50.8362	26.4852	**4.2867**
			0.999	2421.699	**5.698**	638.3646	2345.926	2421.699	1172.073	55.0372
		10	0.8	152.7874	104.4895	43.1458	111.9801	119.0785	74.8448	**7.449**
			0.95	553.3428	212.5212	148.3353	346.9086	541.659	268.4885	**17.3912**
			0.999	26890.13	**36.0226**	7068.979	26047.81	26890.13	13002.6	557.0535
	50	1	0.8	3.0127	2.9799	2.7991	2.9294	2.5421	2.7977	**1.6611**
			0.95	4.0714	3.8165	3.2156	3.6192	3.6355	3.3393	**2.1287**
			0.999	69.0829	**3.6436**	20.3854	63.3417	69.0834	33.6802	5.5786
		3	0.8	6.0401	5.6186	3.5748	6.0356	2.5295	4.1444	**1.8717**

		0.95	14.9977	11.812	6.1234	12.5007	9.4671	8.3812	**2.8433**
		0.999	599.3054	**10.7486**	159.4636	546.7704	599.3068	280.4733	17.3162
	10	0.8	40.2032	35.644	12.7445	40.1489	12.7329	19.7139	**3.665**
		0.95	139.2015	103.8733	38.9681	110.8799	88.6376	65.859	**7.0602**
		0.999	6629.406	**91.2743**	1740.234	6044.489	6629.408	3086.268	141.5039
100	1	0.8	2.9321	2.9219	2.8429	2.8961	2.5434	2.8212	**1.6609**
		0.95	3.4372	3.3549	3.0527	3.2601	3.0723	3.0665	**2.0716**
		0.999	37.6542	**4.1778**	12.3	32.9267	37.653	19.264	4.4078
	3	0.8	4.5006	4.3811	3.2633	4.5006	2.327	3.5231	**1.7609**
		0.95	9.1339	7.8919	4.5997	8.8268	4.4469	5.7377	**2.4232**
		0.999	317.3564	15.6873	87.4449	273.7222	317.3513	151.9645	**11.1389**
	10	0.8	22.3369	21.0644	8.1071	22.3369	7.1352	11.697	**2.7147**
		0.95	74.3299	60.5298	22.1861	70.7494	30.4026	36.442	**4.8928**
		0.999	3500.284	146.8468	943.1045	3014.12	3500.256	1663.128	**76.7617**

Table 7: Summary of performances of Estimators

	N	Estimator	No of the times estimators perform best		
			P3	P7	Total
AM	20	LE	3	3	6
		KL	1	0	1
		MTPAM	5	6	11
	50	LE	3	3	6
		KL	3	0	3
		MTPAM	3	6	9
	100	LE	3	1	4
		KL	3	0	3
		MTPAM	3	8	11
GEN	20	LE	3	3	6
		RE	4	6	10
		KL	2	0	2
	50	LE	3	3	6
		RE	1	3	4
		KL	5	3	8
	100	LE	3	3	6
		RE	0	2	2
		KL	6	4	10
HM	20	LE	3	3	6
		RE	4	6	10
		KL	2	0	2
	50	LE	3	3	6
		RE	1	3	4
		KL	5	3	8
	100	LE	3	3	6
		RE	0	2	2
		KL	6	4	10
GM	20	LE	3	3	6
		RE	4	6	10
		KL	2	0	2
	50	LE	3	3	6

		RE	1	3	4
		KL	5	2	7
		MTPGM	0	1	1
	100	LE	3	3	6
		RE	0	1	1
		KL	6	2	8
		MTPGM	0	3	3
MD	20	LE	3	3	6
		RE	4	0	4
		KL	2	0	2
		MTPMD	0	6	6
	50	LE	3	3	6
		RE	1	0	1
		KL	5	1	6
		MTPMD	0	5	5
	100	LE	3	3	6
		RE	0	0	0
		KL	5	1	6
		MTPMD	1	5	6
MR	20	LE	3	3	6
		KL	1	0	1
		MTPMR	5	6	11
	50	LE	3	3	6
		KL	3	0	3
		MTPMR	3	6	9
	100	LE	3	1	4
		KL	3	0	3
		MTPMR	3	8	11

3.2. Discussions of Result

The simulated results for the three sample sizes (20, 50, and 100), p= 3 and 7 ρ= 0.8, 0.95 and 0.999, σ= 1, 3 and 10. From Tables 1 – 6, it is observed that as σ increases, the MSE also increases. However, MSE decreases as the sample size increases.

For the arithmetic mean version of k, when p =3 and 7, MTPAM performs overall best, having the minimum MSE across all the sample sizes, multicollinearity, and sigma levels. The L.E. and K.L. also dominate on some occasions. Table 7 reveals that the K.L. estimator performs best seven (7) times, L.E. sixteen (16) times, and MTPAM thirty-one (31) times. The OLS estimator performed worst in all, with the highest value of MSE.

For the generalized k, when p =3 and 7, MTGEN did not perform well considering its MSE across all the sample sizes, multicollinearity, and sigma levels. The L.E., RE, and K.L. dominate other estimators. Table 7 shows that the K.L estimator performs best twenty (20) times, L.E. eighteen (18) times, and RE sixteen (16) times. The OLS estimator performed worst in all, with the highest value of MSE.

For the harmonic mean version of k, when p =3 and 7, the L.E., RE, and K.L. dominate other estimators. Results from Table 7 reveal that the K.L. estimator performs best twenty (20) times, L.E. eighteen (18) times, and RE sixteen (16) times.

For the geometric mean version of k, when p =3 and 7, MTPGM performs overall best just a few times. The L.E., RE, and K.L. dominate other estimators. Results from Table 7 reveal that the K.L. estimator performs best seventeen (17) times, L.E. eighteen (18) times, RE fifteen (15) times, and MTPGM four (4) times. The OLS estimator did not perform well in this case too.

For the median version of k, when p =3 and 7, L.E. performs best, closely followed by MTPAM, K.L., and lastly, RE. Results from Table 7 reveal that the L.E. performs best eighteen (18) times, MTPMED seventeen (17) times, K.L. fourteen (14) times, and then RE five (5) times.

For the mid-range version of k, when p =3 and 7, MTPMR performs overall best, followed by L.E., and then K.L. From Table 7, it is clearly observed that the MTPMR performs best thirty-one (31) times, L.E. sixteen (17) times, and then K.L. seven (7) times.

4. Conclusion

This article proposes a modified two-parameter estimator for estimating the linear regression models parameter when regressors are correlated. The proposed modified two-parameter estimator uses the generalized biasing parameter *k* and other forms of *k* like the arithmetic mean, harmonic mean, geometric mean, median, and mid-range.

The performance of the proposed estimator with some existing ones in terms of the mean squared error criterion was evaluated. The proposed estimator $\hat{\alpha}_{MTPE}(k, d, b)$ performs better than the other five estimators at the different sample sizes, sigma, multicollinearity levels, and the number of parameters when the biasing parameter k is the arithmetic mean and mid-range versions. However, the L.E., RE, and K.L. dominate other estimators when the versions of the biasing parameter *k* are generalized, harmonic, and geometric. The proposed estimator $\hat{\alpha}_{MTPE}(k, d, b)$ and the LE are the leading estimators in terms of the performance of the MSE when the median version of the biasing parameter *k* is used. The arithmetic mean and mid-range versions of the biasing parameter *k* are recommended for the proposed estimator to combat the problem of multicollinearity in the linear regression model. For further studies, the almost unbiased and unbiased estimator of the proposed estimator $\hat{\alpha}_{MTPE}(k, d, b)$ can be proposed.

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