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Lie symmetry analysis and exact solutions of conformable time-space fractional Benjamin-Bona-Mahony equations

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Abstract

This article investigates the invariance properties of nonlinear fractional partial differential equations involving conformable fractional derivatives in both time and space through the framework of Lie symmetry analysis. The study focuses on three significant forms of the Benjamin-Bona-Mahony (BBM) equations. The corresponding Lie symmetry generators and vector fields are systematically derived and employed to reduce the governing equations to ordinary differential equations. Exact analytical solutions are then constructed by utilising the (G'/G)-expansion method for the time-space fractional BBM equation. At the same time, suitable ansatz-based techniques are adopted to obtain explicit solutions for the modified and power-law fractional BBM equations. Furthermore, a qualitative dynamical analysis is carried out for the time-space fractional BBM equation and its modified form using bifurcation analysis and phase portraits, which reveal the behaviour of solution trajectories near the critical equilibrium points and provide insight into the nonlinear dynamics of the considered models. The results demonstrate that combining Lie symmetry analysis with analytical solution techniques provides an effective, systematic approach to studying nonlinear fractional BBM-type equations and deepens understanding of their mathematical structure and physical characteristics.

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1. Introduction

Fractional calculus has attracted considerable attention over recent decades for addressing a variety of scientific problems, leading to significant theoretical advances. Several forms of fractional derivatives, including the Riemann–Liouville and Riesz derivatives, have been introduced to provide more accurate and realistic descriptions of physical processes. The Lie symmetry technique has also been extended to conformable differential equations for solving space-time fractional nonlinear partial differential equations

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(NLPDEs), while retaining derivative properties such as the chain rule. The conformable time and space fractional derivatives are defined, respectively, as follows [1–5]:

$$D^{\beta}[\Lambda(t)] = \lim_{\epsilon\to 0}\frac{\Lambda(t+\epsilon t^{1-\beta})-\Lambda(t)}{\epsilon}, \tag{1}$$

$$D^{\beta}[\Lambda(x)] = \lim_{\epsilon\to 0}\frac{\Lambda(x+\epsilon x^{1-\beta})-\Lambda(x)}{\epsilon}, \tag{2}$$

for each $t, x > 0$, where $\beta \in (0, 1]$. Suppose that the conformable fractional derivatives $D^{\beta}[\Lambda(t)]$ and $D^{\beta}[\Lambda(x)]$ are well defined for t and x, respectively, in $(0, b)$, with $b > 0$.

Researchers have developed numerous variants of the Benjamin–Bona–Mahony (BBM) equation because of its rich dynamics and broad range of applications. This well-known model describes the propagation of surface water waves in a uniform channel and has also been used for long-wavelength surface waves in liquids, acoustic waves in harmonic crystals, hydromagnetic waves in cold plasma, and acoustic-gravity waves in compressible fluids. This study focuses on three distinct time-space fractional BBM models.

The single-nonlinearity time-space fractional BBM equation is [6]:

$$T_t^{\alpha}(u) + uT_x^{\alpha}(u) - T_t^{\alpha}(T_x^{\alpha}(T_x^{\alpha}(u))) = 0. \tag{3}$$

The modified time-space fractional BBM equation is [6–9]:

$$T_t^{\alpha}(u) + u^2T_x^{\alpha}(u) - T_t^{\alpha}(T_x^{\alpha}(T_x^{\alpha}(u))) = 0. \tag{4}$$

The power-law time-space fractional BBM equation is [6, 10]:

$$T_t^{\alpha}(u) + u^nT_x^{\alpha}(u) - T_t^{\alpha}(T_x^{\alpha}(T_x^{\alpha}(u))) = 0. \tag{5}$$

Here, $u = u(x, t)$, and

$$T_t^{\alpha}(u) = t^{1-\alpha}\frac{\partial u}{\partial t}, \quad T_x^{\alpha}(u) = x^{1-\alpha}\frac{\partial u}{\partial x}, \tag{6}$$

$$T_t^{\alpha}(T_x^{\alpha}(T_x^{\alpha}(u))) = (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\frac{\partial^2 u}{\partial t\partial x} + t^{1-\alpha}x^{2-2\alpha}\frac{\partial^3 u}{\partial t\partial x^2}. \tag{7}$$

Also, $\alpha \in (0, 1]$.

Accordingly, effective analytical and numerical techniques have been developed to obtain solutions to these equations. Relevant methods include the exp-function scheme [11, 12], the Kudryashov method [13, 14], the sine–cosine method [15], the homotopy perturbation method [6], and the Riccati equation technique [16, 17]. Explicit solutions for these models have also been constructed using Lie symmetry methods and the (G'/G)-expansion method [18–21]. Numerical methods are likewise used to obtain approximate solutions when exact solutions are difficult or impossible to derive [22–24], providing insight into the dynamical behaviour of nonlinear models and complex physical phenomena.

The remainder of this paper is organised as follows. Section 2 determines the symmetries of the three equations using conformable fractional Lie symmetry analysis. Section 3 applies the (G'/G)-expansion and ansatz methods to obtain explicit exact solutions. Section 4 presents a bifurcation analysis of two reduced ordinary differential equations (ODEs), followed by the Conclusion.

2. Conformable fractional Lie symmetry analysis

First, the Lie group of point transformations is introduced [25–28]:

$$x^* = x + \epsilon\,\xi(x, t, u(x, t)) + O(\epsilon^2), \tag{8}$$

$$t^* = t + \epsilon\,\tau(x, t, u(x, t)) + O(\epsilon^2), \tag{9}$$

$$u^*(x^*, t^*) = u + \epsilon\,\eta(x, t, u(x, t)) + O(\epsilon^2). \tag{10}$$

where $\epsilon \ll 1$ is a free parameter. The Lie invariance criterion in prolonged form is

$$Pr^{(3)}V(E)|_{E=0} = 0, \tag{11}$$

which is appropriate for the highest-order derivatives appearing in the governing equation.

For completeness, the conformable fractional prolongation used to derive the determining equations is stated explicitly. The prolonged coefficients associated with the conformable derivatives are

$$\eta_t^{\alpha} = t^{1-\alpha}\eta_t + (1-\alpha)\tau t^{-\alpha}u_t, \tag{12}$$

$$\eta_x^{\alpha} = x^{1-\alpha}\eta_x + (1-\alpha)\xi x^{-\alpha}u_x, \tag{13}$$

and

$$\begin{aligned}\eta_{xxt}^{3\alpha} &= (1-\alpha)(t^{1-\alpha}x^{1-2\alpha}\eta_{xt} + (1-2\alpha)t^{1-\alpha}x^{-2\alpha}\xi u_{xt} + (1-\alpha)t^{-\alpha}\tau x^{1-2\alpha}u_{xt}) \\ &\quad + t^{1-\alpha}x^{2-2\alpha}\eta_{xxt} + (2-2\alpha)x^{1-2\alpha}t^{1-\alpha}\xi u_{xxt} + (1-\alpha)t^{-\alpha}x^{2-2\alpha}\tau u_{xxt}.\end{aligned} \tag{14}$$

Here, $\eta_t, \eta_x, \eta_{xt}$, and η_{xxt} denote the corresponding ordinary prolongation coefficients of the infinitesimal generator X, as defined in Section 3, Eqs. (11)–(12) of Refs. [27, 28]. Substitution of these ordinary prolongation coefficients into the conformable prolongation relations above gives the coefficients used in the present determining equations.

2.1. Single-nonlinearity time-space fractional BBM equation

Under the point transformations in Eqs. (8)–(10), u^* is a solution whenever u is a solution of Eq. (3). The invariance condition for Eq. (3) is

$$\eta_t^{\alpha} + u\eta_x^{\alpha} + u_x^{\alpha}\eta - \eta_{xxt}^{3\alpha} = 0. \tag{15}$$

Substituting the extended infinitesimals η, η_t^{α}, η_x^{α}, and $\eta_{xxt}^{3\alpha}$ and equating the coefficients of the resulting derivatives of u to zero yields the following overdetermined system of linear partial differential equations:

$$\xi_t = \xi_u = \tau_x = \tau_u = \eta_{uu} = \eta_{tu} = 0, \tag{16}$$

$$t^{1-\alpha}\eta_t + x^{1-\alpha}u\eta_x - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\eta_{xt} - t^{1-\alpha}x^{2-2\alpha}\eta_{xxt} = 0, \tag{17}$$

$$x^{1-\alpha}u\tau_t - (1-\alpha)x^{1-\alpha}t^{-1}u\tau + x^{1-\alpha}u\xi_x - (1-\alpha)x^{-\alpha}u\xi + x^{1-\alpha}\eta = 0, \tag{18}$$

$$2t^{1-\alpha}\xi_x - 2(1-\alpha)t^{1-\alpha}x^{-1}\xi - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\eta_{xu} - t^{1-\alpha}x^{2-2\alpha}\eta_{xxu} = 0, \tag{19}$$

$$t^{1-\alpha}x^{2-2\alpha}\xi_{xx} - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\xi_x + (1-\alpha)t^{1-\alpha}x^{-2\alpha}\xi - 2t^{1-\alpha}x^{2-2\alpha}\eta_{xu} = 0. \tag{20}$$

Solving this system gives the general solution

$$\eta = k_1 u, \qquad \xi = k_2 x^{1-\alpha}, \qquad \tau = -\frac{k_1}{\alpha}t + k_3 t^{1-\alpha}. \tag{21}$$

Here, k_1, k_2, and k_3 are arbitrary constants. The corresponding generators of the associated vector fields are

$$V_1 = -\frac{t}{\alpha}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \qquad V_2 = x^{1-\alpha}\frac{\partial}{\partial x}, \qquad V_3 = t^{1-\alpha}\frac{\partial}{\partial t}. \tag{22}$$

To derive the symmetry reduction of Eq. (3), the characteristic equation for the infinitesimal generator is

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \tag{23}$$

where ξ, τ, and η are given by Eq. (21). Substituting the corresponding similarity solution into Eq. (3) yields the reduced ODE.

The vector field V_1 provides the following similarity variables:

$$\hat{\zeta} = x, \qquad u = t^{-\alpha}H\left(\hat{\zeta}\right). \tag{24}$$

Substituting Eq. (24) into Eq. (3) gives the corresponding ODE

$$-\alpha H\left(\hat{\zeta}\right) + \hat{\zeta}^{1-\alpha}H\left(\hat{\zeta}\right)H'\left(\hat{\zeta}\right) + \alpha(1-\alpha)\hat{\zeta}^{1-2\alpha}H'\left(\hat{\zeta}\right) + \alpha\hat{\zeta}^{2-2\alpha}H''\left(\hat{\zeta}\right) = 0. \tag{25}$$

The vector field $V_2 + V_3$ provides the similarity variables

$$\hat{\zeta} = \frac{x^{\alpha}}{\alpha} - \frac{t^{\alpha}}{\alpha}, \qquad u = H\left(\hat{\zeta}\right). \tag{26}$$

Substituting Eq. (26) into Eq. (3) gives

$$-H'\left(\hat{\zeta}\right) + H'\left(\hat{\zeta}\right)H\left(\hat{\zeta}\right) + H'''\left(\hat{\zeta}\right) = 0. \tag{27}$$

Integrating Eq. (27) with respect to $\hat{\zeta}$ introduces an arbitrary integration constant. Because this work focuses on localized travelling-wave solutions constructed using the (G'/G)-expansion method, the standard decay conditions $H, H', H'' \to 0$ as $|\hat{\zeta}| \to \infty$ are imposed. These conditions imply that the integration constant vanishes, $C = 0$, giving

$$2H''\left(\hat{\zeta}\right) + H\left(\hat{\zeta}\right)^2 - 2H\left(\hat{\zeta}\right) = 0. \tag{28}$$

2.2. Modified time-space fractional BBM equation

Under the point transformations in Eqs. (8)–(10), u^* is a solution whenever u is a solution of Eq. (4). The invariance condition for Eq. (4) is

$$\eta_t^{\alpha} + u^2\eta_x^{\alpha} + 2uu_x^{\alpha}\eta - \eta_{xxt}^{3\alpha} = 0. \tag{29}$$

Substituting the extended infinitesimals η_t^{α}, η_x^{α}, and $\eta_{xxt}^{3\alpha}$ and equating the coefficients of the resulting derivatives of u to zero yields the following overdetermined system of linear partial differential equations.

$$\xi_t = \xi_u = \tau_x = \tau_u = \eta_{uu} = \eta_{tu} = 0, \tag{30}$$

$$t^{1-\alpha}\eta_t + x^{1-\alpha}u^2\eta_x - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\eta_{xt} - t^{1-\alpha}x^{2-2\alpha}\eta_{xxt} = 0, \tag{31}$$

$$2t^{1-\alpha}\xi_x - 2(1-\alpha)t^{1-\alpha}x^{-1}\xi - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\eta_{xu} - t^{1-\alpha}x^{2-2\alpha}\eta_{xxu} = 0, \tag{32}$$

$$x^{1-\alpha}u^2\tau_t - (1-\alpha)x^{1-\alpha}t^{-1}u^2\tau + x^{1-\alpha}u^2\xi_x - (1-\alpha)x^{-\alpha}u^2\xi + 2x^{1-\alpha}u\eta = 0, \tag{33}$$

$$t^{1-\alpha}x^{2-2\alpha}\xi_{xx} - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\xi_x + (1-\alpha)t^{1-\alpha}x^{-2\alpha}\xi - 2t^{1-\alpha}x^{2-2\alpha}\eta_{xu} = 0. \tag{34}$$

Solving this system gives

$$\eta = uk_1, \qquad \xi = k_2x^{1-\alpha}, \qquad \tau = -\frac{2k_1}{\alpha}t + k_3t^{1-\alpha}. \tag{35}$$

Here, k_1, k_2, and k_3 are arbitrary constants. The corresponding generators of the associated vector fields are

$$V_1 = -\frac{2t}{\alpha}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \qquad V_2 = x^{1-\alpha}\frac{\partial}{\partial x}, \qquad V_3 = t^{1-\alpha}\frac{\partial}{\partial t}. \tag{36}$$

To derive the symmetry reduction of Eq. (4), the characteristic equation for the infinitesimal generator is

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \tag{37}$$

where ξ, τ, and η are given by Eq. (35). Substituting the solution associated with Eqs. (8)–(10) into Eq. (4) yields the reduced ODE.

The vector field V_1 gives the similarity variables

$$\hat{\zeta} = x, \; u = t^{-\frac{\alpha}{2}}H\left(\hat{\zeta}\right). \tag{38}$$

Substituting Eq. (38) into Eq. (4) gives

$$-\frac{\alpha}{2}H\left(\hat{\zeta}\right) + \hat{\zeta}^{1-\alpha}H\left(\hat{\zeta}\right)^2H'\left(\hat{\zeta}\right) + \frac{\alpha(1-\alpha)}{2}\hat{\zeta}^{1-2\alpha}H'\left(\hat{\zeta}\right) + \frac{\alpha}{2}\hat{\zeta}^{2-2\alpha}H''\left(\hat{\zeta}\right) = 0. \tag{39}$$

The vector field $V_2 + V_3$ provides the similarity variables

$$\hat{\zeta} = \frac{x^{\alpha}}{\alpha} - \frac{t^{\alpha}}{\alpha}, \qquad u = H\left(\hat{\zeta}\right). \tag{40}$$

Substituting Eq. (40) into Eq. (4) gives

$$-H'\left(\hat{\zeta}\right) + H'\left(\hat{\zeta}\right)H\left(\hat{\zeta}\right)^2 + H'''\left(\hat{\zeta}\right) = 0. \tag{41}$$

Integrating Eq. (41) with respect to $\hat{\zeta}$ introduces an arbitrary integration constant. Because this work focuses on localized homoclinic solutions constructed using the sech-expansion method, the standard decay conditions $H, H', H'' \to 0$ as $|\hat{\zeta}| \to \infty$ are imposed. These conditions imply that the integration constant vanishes, $C = 0$, giving

$$3H''\left(\hat{\zeta}\right) + H\left(\hat{\zeta}\right)^3 - 3H\left(\hat{\zeta}\right) = 0. \tag{42}$$

2.3. Power-law time-space fractional BBM equation

Under the point transformations in Eqs. (8)–(10), u^* is a solution whenever u is a solution of Eq. (5). The invariance condition for Eq. (5) is

$$\eta_t^{\alpha} + u^n \eta_x^{\alpha} + n u^{n-1} u_x^{\alpha} \eta - \eta_{xxt}^{3\alpha} = 0. \tag{43}$$

Substituting the extended infinitesimals η_t^{α}, η_x^{α}, and $\eta_{xxt}^{3\alpha}$ and equating the coefficients of the resulting derivatives of u to zero yields the following overdetermined system of linear partial differential equations:

$$\xi_t = \xi_u = \tau_x = \tau_u = \eta_{uu} = \eta_{tu} = 0, \tag{44}$$

$$t^{1-\alpha}\eta_t + x^{1-\alpha}u^n\eta_x - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\eta_{xt} - t^{1-\alpha}x^{2-2\alpha}\eta_{xxt} = 0, \tag{45}$$

$$2t^{1-\alpha}\xi_x - 2(1-\alpha)t^{1-\alpha}x^{-1}\xi - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\eta_{xu} - t^{1-\alpha}x^{2-2\alpha}\eta_{xxu} = 0, \tag{46}$$

$$x^{1-\alpha}u^n\tau_t - (1-\alpha)x^{1-\alpha}t^{-1}u^n\tau + x^{1-\alpha}u^n\xi_x - (1-\alpha)x^{-\alpha}u^n\xi + nx^{1-\alpha}u^{n-1}\eta = 0, \tag{47}$$

$$t^{1-\alpha}x^{2-2\alpha}\xi_{xx} - (1-\alpha)t^{1-\alpha}x^{1-2\alpha}\xi_x + (1-\alpha)t^{1-\alpha}x^{-2\alpha}\xi - 2t^{1-\alpha}x^{2-2\alpha}\eta_{xu} = 0. \tag{48}$$

Solving this system gives

$$\eta = k_1 u, \qquad \xi = k_2 x^{1-\alpha}, \qquad \tau = -\frac{nk_1}{\alpha}t + k_3 t^{1-\alpha}. \tag{49}$$

Here, k_1, k_2, and k_3 are arbitrary constants. The corresponding generators of the associated vector fields are

$$V_1 = -\frac{nt}{\alpha}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \qquad V_2 = x^{1-\alpha}\frac{\partial}{\partial x}, \qquad V_3 = t^{1-\alpha}\frac{\partial}{\partial t}. \tag{50}$$

To derive the symmetry reduction of Eq. (5), the characteristic equation for the infinitesimal generator is

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \tag{51}$$

where ξ, τ, and η are given by Eq. (49). Substituting the corresponding similarity solution into Eq. (5) yields the reduced ODE.

The vector field V_1 gives the following similarity variables:

$$\hat{\zeta} = x, \qquad u = t^{-\frac{\alpha}{n}} H\left(\hat{\zeta}\right). \tag{52}$$

Substituting Eq. (52) into Eq. (5) gives

$$-\frac{\alpha}{n}H\left(\hat{\zeta}\right) + \hat{\zeta}^{1-\alpha}H\left(\hat{\zeta}\right)^n H'\left(\hat{\zeta}\right) + \frac{\alpha(1-\alpha)}{n}\hat{\zeta}^{1-2\alpha}H'\left(\hat{\zeta}\right) + \frac{\alpha}{n}\hat{\zeta}^{2-2\alpha}H''\left(\hat{\zeta}\right) = 0. \tag{53}$$

The vector field $V_2 + V_3$ provides the similarity variables

$$\hat{\zeta} = \frac{x^{\alpha}}{\alpha} - \frac{t^{\alpha}}{\alpha}, \qquad u = H\left(\hat{\zeta}\right). \tag{54}$$

Substituting Eq. (54) into Eq. (5) gives

$$-H'\left(\hat{\zeta}\right) + H'\left(\hat{\zeta}\right)H\left(\hat{\zeta}\right)^n + H'''\left(\hat{\zeta}\right) = 0. \tag{55}$$

Integrating Eq. (55) with respect to $\hat{\zeta}$ introduces an arbitrary integration constant. Because this work focuses on localized homoclinic solutions constructed using the sech-expansion method, the standard decay conditions $H, H', H'' \to 0$ as $|\hat{\zeta}| \to \infty$ are imposed. These conditions imply that the integration constant vanishes, $C = 0$, giving

$$(n+1)H''\left(\hat{\zeta}\right) + H\left(\hat{\zeta}\right)^{n+1} - (n+1)H\left(\hat{\zeta}\right) = 0. \tag{56}$$

3. Exact solutions of reduced ODEs

This section derives exact analytical solutions for the time-space fractional BBM models in Eqs. (3)–(5). The (G'/G)-expansion method [18–20] is used to obtain explicit solutions of Eq. (3). This method systematically constructs exact travelling-wave solutions by transforming the governing equation into a solvable ordinary differential equation.

For the modified time-space fractional BBM Eq. (4) and the power-law time-space fractional BBM Eq. (5), ansatz-based techniques are used to derive exact solutions. These methods are selected according to the nonlinear structure of the corresponding equations and permit the construction of analytical solutions. Together, the (G'/G)-expansion method and the ansatz approaches provide effective analytical techniques for investigating nonlinear fractional BBM-type equations.

3.1. Solution of Eq. (3)

To solve Eq. (28), assume that the ODE admits a solution expressed in powers of (G'/G):

$$H\left(\hat{\zeta}\right) = e_m \left(G'/G\right)^m + e_{m-1}\left(G'/G\right)^{m-1} + \cdots + e_0, \tag{57}$$

where e_m are arbitrary constants, $m = 0, 1, 2, 3, \ldots$, with $e_m \neq 0$. The function $G(\hat{\zeta})$ satisfies the second-order linear differential equation

$$G'' + \tilde{\omega}G' + \tilde{\delta}G = 0. \tag{58}$$

Here, $\tilde{\omega}$ and $\tilde{\delta}$ are arbitrary constants. For Eq. (28), take $H(\hat{\zeta}) \sim \hat{\zeta}^{-m}$, so that $H''(\hat{\zeta}) \sim \hat{\zeta}^{-m-2}$ and $H^2(\hat{\zeta}) \sim \hat{\zeta}^{-2m}$. Balancing the dominant terms H'' and H^2 gives $m + 2 = 2m$, and hence $m = 2$. Therefore, the ansatz in Eq. (57) becomes

$$H\left(\hat{\zeta}\right) = e_0 + e_1\left(G'/G\right) + e_2\left(G'/G\right)^2, e_2 \neq 0. \tag{59}$$

From Eq. (59), we obtain

$$H\left(\hat{\zeta}\right)^2 = {e_0}^2 + 2\,e_0 e_1\left(G'/G\right) + 2\,e_0 e_2\left(G'/G\right)^2 + {e_1}^2\left(G'/G\right)^2 + 2\,e_1 e_2\left(G'/G\right)^3 + {e_2}^2\left(G'/G\right)^4, \tag{60}$$

$$\begin{aligned} H''\left(\hat{\zeta}\right) = {} & \left(e_1\tilde{\omega}\tilde{\delta} + 2e_2\tilde{\delta}^2\right)\left(G'/G\right)^0 \\ & + \left(e_1\tilde{\omega}^2 + 2e_1\tilde{\delta} + 6e_2\tilde{\delta}\tilde{\omega}\right)\left(G'/G\right)^1 \\ & + \left(3e_1\tilde{\omega} + 4e_2\tilde{\omega}^2 + 8e_2\tilde{\delta}\right)\left(G'/G\right)^2 \\ & + \left(2e_1 + 10e_2\tilde{\omega}\right)\left(G'/G\right)^3 + 6e_2\left(G'/G\right)^4, \end{aligned} \tag{61}$$

Substituting Eq. (59) into Eq. (28) gives a polynomial in (G'/G). Equating the coefficients of like powers of (G'/G) to zero yields

$$\left(G'/G\right)^4 : \frac{1}{2}e_2^2 + 6e_2, \tag{62}$$

$$\left(G'/G\right)^3 : e_1 e_2 + 2e_1 + 10e_2\tilde{\omega}, \tag{63}$$

$$\left(G'/G\right)^2 : 3e_1\tilde{\omega} + e_0 e_2 - e_2 + 8e_2\tilde{\delta} + 4e_2\tilde{\omega}^2 + \frac{1}{2}e_1^2, \tag{64}$$

$$\left(G'/G\right) : 6e_2\tilde{\delta}\tilde{\omega} + e_0 e_1 + e_1\tilde{\omega}^2 + 2e_1\tilde{\delta} - e_1, \tag{65}$$

$$\left(G'/G\right)^0 : -e_0 + 2e_2\tilde{\delta}^2 + \frac{1}{2}e_0^2 + e_1\tilde{\omega}\tilde{\delta}, \tag{66}$$

Solving Eqs. (62)–(66) yields the following two parameter sets. For set 1,

$$\tilde{\delta} = \frac{1}{576}a^2 - \frac{1}{4}, \tilde{\omega} = -\frac{1}{12}a, e_0 = 3 - \frac{1}{48}a^2, e_1 = a, e_2 = -12. \tag{67}$$

For set 2,

$$\tilde{\delta} = \frac{1}{576}a^2 + \frac{1}{4}, \tilde{\omega} = -\frac{1}{12}a, e_0 = -1 - \frac{1}{48}a^2, e_1 = a, e_2 = -12. \tag{68}$$

Here, a is an arbitrary constant. Substituting set 1 into the ansatz in Eq. (59) gives

$$H\left(\hat{\zeta}\right) = 3 - \frac{1}{48}a^2 + a\left(G'/G\right) - 12\left(G'/G\right)^2, \tag{69}$$

For the exact solutions, let Let $q(\hat{\zeta}) = G'/G$, where G satisfies Eq. (58). For $\Delta = \tilde{\omega}^2 - 4\tilde{\delta} > 0$, the auxiliary equation has the solution

$$G\left(\hat{\zeta}\right) = e^{-\frac{\tilde{\omega}}{2}\hat{\zeta}}\left(J_1\cosh\left(\frac{\sqrt{\Delta}}{2}\hat{\zeta}\right) + J_2\sinh\left(\frac{\sqrt{\Delta}}{2}\hat{\zeta}\right)\right). \tag{70}$$

Hence,

$$\frac{G'}{G} = -\frac{\tilde{\omega}}{2} + \frac{\sqrt{\Delta}}{2}\frac{J_1\sinh\left(\frac{\sqrt{\Delta}}{2}\hat{\zeta}\right) + J_2\cosh\left(\frac{\sqrt{\Delta}}{2}\hat{\zeta}\right)}{J_1\cosh\left(\frac{\sqrt{\Delta}}{2}\hat{\zeta}\right) + J_2\sinh\left(\frac{\sqrt{\Delta}}{2}\hat{\zeta}\right)}. \tag{71}$$

The expanded solution is obtained as follows. If $\tilde{\omega}^2 - 4\tilde{\delta} > 0$, the exact solution of Eq. (3) is

$$u(x,t) = \left(\frac{a\sqrt{\tilde{\omega}^2 - 4\tilde{\delta}}\left(J_1 \sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)}{2\left(J_1 \cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)} - \frac{\tilde{\omega}}{2}a \right) \right.$$
$$\left. - 12\left(\frac{\sqrt{\tilde{\omega}^2 - 4\tilde{\delta}}\left(J_1 \sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)}{2\left(J_1 \cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)} - \frac{\tilde{\omega}}{2} \right)^2 + 3 - \frac{1}{48}a^2, \tag{72}$$

where J_1 and J_2 are arbitrary constants. Substituting set 2 into the ansatz in Eq. (59) gives

$$H\left(\hat{\zeta}\right) = -1 - \frac{1}{48}a^2 + a\left(G'/G\right) - 12\left(G'/G\right)^2. \tag{73}$$

If $\tilde{\omega}^2 - 4\tilde{\delta} < 0$, the corresponding solution of Eq. (3) is

$$u(x,t) = \frac{a\sqrt{-\tilde{\omega}^2 + 4\tilde{\delta}}\left(J_1 \sin\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \cos\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)}{2\left(J_1 \cos\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \sin\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)} - \frac{\tilde{\omega}}{2}a$$
$$+ 12\left(\frac{\sqrt{-\tilde{\omega}^2 + 4\tilde{\delta}}\left(J_1 \sin\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \cos\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)}{2\left(J_1 \cos\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right) + J_2 \sin\left(\frac{\sqrt{-\tilde{\omega}^2+4\tilde{\delta}}\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)}{2}\right)\right)} - \frac{\tilde{\omega}}{2} \right)^2 - 1 - \frac{1}{48}a^2, \tag{74}$$

where J_1 and J_2 are arbitrary constants.

Figure 1 shows the three-dimensional and corresponding two-dimensional representations of the kink-soliton solution $u(x,t)$. The solution exhibits a smooth, continuous transition between two distinct asymptotic states, and the two-dimensional profile illustrates its characteristic monotonic transition.

Figure 2 shows the three-dimensional and corresponding two-dimensional representations of the singular periodic solution $u(x,t)$. The solution exhibits a periodic wave structure containing singularities, while the two-dimensional profile shows the periodic oscillations and singular behaviour.

3.2. *Solution of Eq. (4)*

Assume that the ODE in Eq. (42) admits a solution of the form

$$H\left(\hat{\zeta}\right) = A_0 + A_1 \operatorname{sech}\left(\kappa\hat{\zeta}\right) + \cdots + A_p \operatorname{sech}\left(\kappa\hat{\zeta}\right)^p, \tag{75}$$

where A_i, $i = 0, 1, 2, \ldots, p$, are arbitrary constants with $A_p \neq 0$. Balancing H'' and H^3 in Eq. (42) gives $p + 2 = 3p$, so $p = 1$. Therefore, the ansatz becomes

$$H\left(\hat{\zeta}\right) = A_0 + A_1 \operatorname{sech}\left(\kappa\hat{\zeta}\right). \tag{76}$$

Substituting Eq. (76) into Eq. (42) gives a polynomial in $\operatorname{sech}(\kappa\hat{\zeta})$. Equating coefficients of like powers and solving the resulting algebraic system gives

$$A_0 = 0, \quad A_1 = \pm\sqrt{6}, \quad \kappa = \pm 1. \tag{77}$$

Substituting these values into Eq. (76) and then substituting the result into Eq. (40) gives the solution of Eq. (4):

$$u(x,t) = \pm\sqrt{6}\operatorname{sech}\left(\frac{x^\alpha}{\alpha} - \frac{t^\alpha}{\alpha}\right). \tag{78}$$

Figure 3 shows the three-dimensional and corresponding two-dimensional representations of the bright-soliton solution $u(x,t)$. The surface and profile show a localised wave with a pronounced amplitude above the background level and decay away from the central peak, confirming the characteristic bright-soliton behaviour.

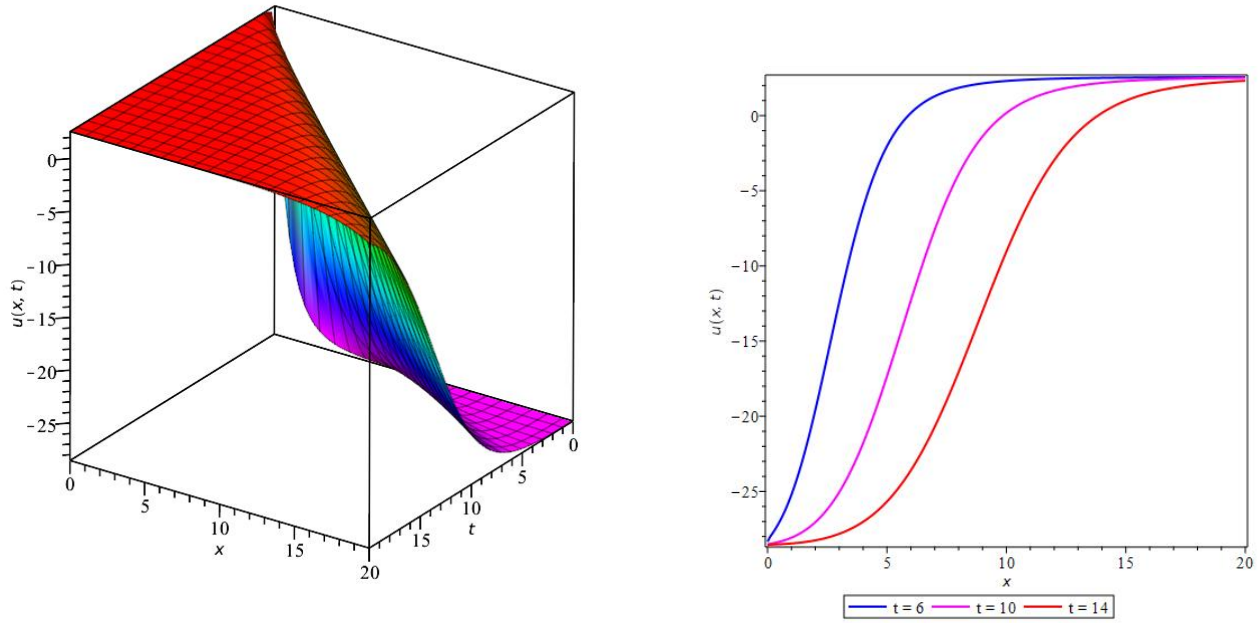


Figure 1: Kink-wave behaviour of the solution in Eq. (72) for $e_1 = -2$, $e_2 = -12$, $J_1 = 5$, $J_2 = 3$, and $\alpha = 0.5$.

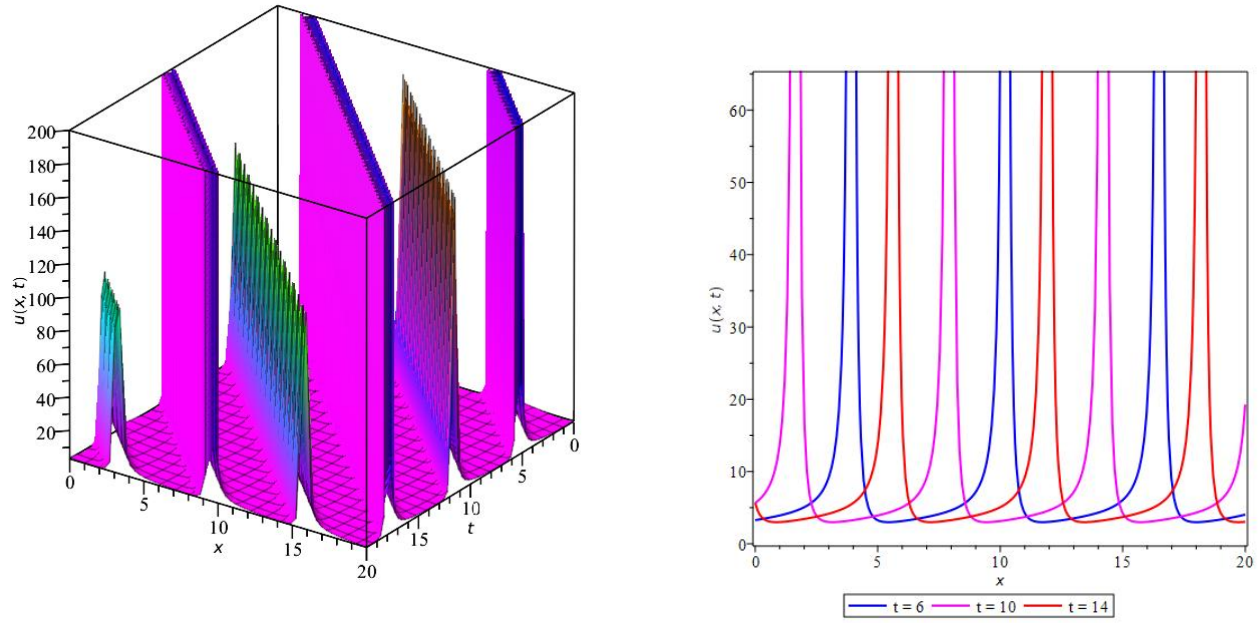


Figure 2: Singular periodic-wave behaviour of the solution in Eq. (74) for $e_1 = 4$, $e_2 = -12$, $J_1 = 5$, $J_2 = 3$, and $\alpha = 1$.

3.3. Solution of Eq. (5)

Assume that the ODE in Eq. (56) admits a solution of the form

$$H\left(\hat{\zeta}\right) = A\text{sech}\left(\kappa\hat{\zeta}\right)^{p}, \tag{79}$$

where $A \neq 0$ is an arbitrary constant. In Eq. (56), the dominant terms are H'' and H^{n+1}. Thus, $p + 2 = (n + 1)p$, which gives $p = 2/n$.

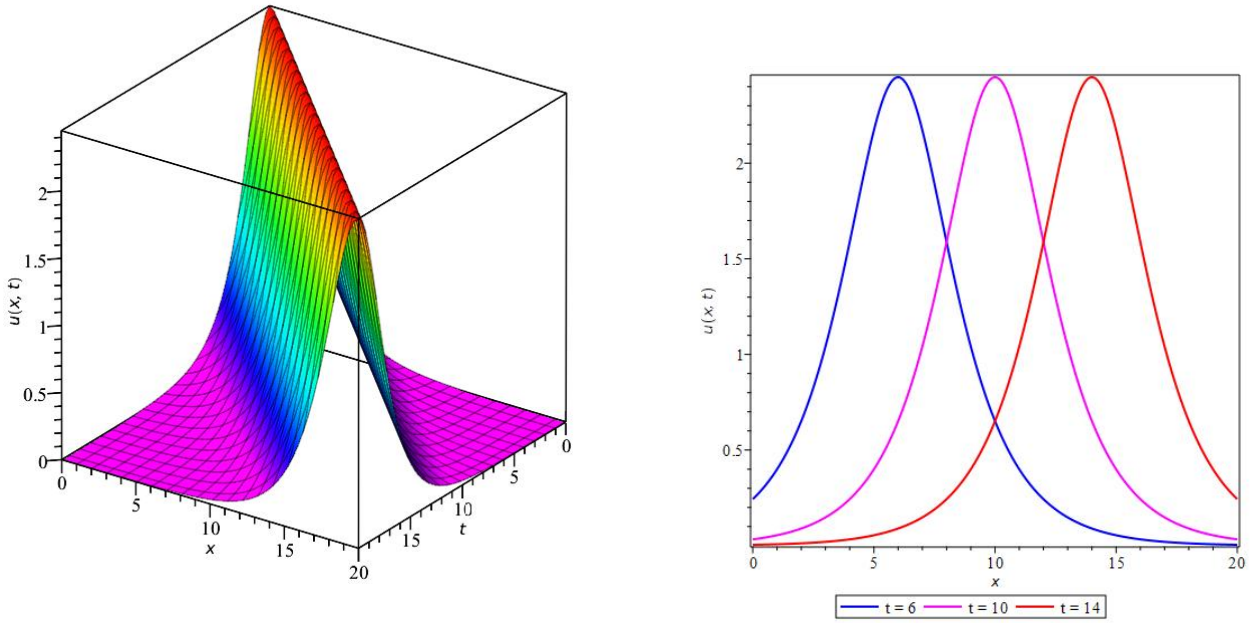


Figure 3: Representation of the bright-soliton solution in Eq. (78) for $\alpha = 1$.

Therefore, the ansatz becomes

$$H\left(\hat{\zeta}\right) = A\text{sech}\left(\kappa\hat{\zeta}\right)^{\frac{2}{n}}. \tag{80}$$

Substituting Eq. (80) into Eq. (56) gives a polynomial in sech. Equating coefficients of like powers and solving the resulting algebraic system gives

$$A = \left(\frac{(n+2)(n+1)}{2}\right)^{\frac{1}{n}}, \quad \kappa = \pm\frac{n}{2}. \tag{81}$$

Substituting these values into Eq. (80) and then substituting the result into Eq. (54) gives the solution of Eq. (5):

$$u(x,t) = \left(\frac{(n+2)(n+1)}{2}\right)^{\frac{1}{n}} \text{sech}\left(\frac{n}{2}\left(\frac{x^{\alpha}}{\alpha} - \frac{t^{\alpha}}{\alpha}\right)\right)^{\frac{2}{n}}, \tag{82}$$

where n is a natural number.

Figure 4 shows the three-dimensional and corresponding two-dimensional representations of the bright-soliton solution $u(x,t)$. The solution forms a smooth, well-localised wave profile whose maximum amplitude is concentrated near the central region and decays towards the background in the far field. The two-dimensional profile further illustrates the localised nature and propagation characteristics of the bright soliton.

4. Bifurcation analysis of reduced ODEs

This section presents the bifurcation analysis [29–31] of the nonlinear autonomous systems associated with Eqs. (28) and (42). The critical points are identified, and their nature and stability are determined using the Jacobian matrix, eigenvalue analysis, and the corresponding first integral.

4.1. Bifurcation analysis corresponding to Eq. (28)

Eq. (28) can be expressed as the following nonlinear autonomous system:

$$H'(\hat{\zeta}) = G(\hat{\zeta}), \qquad G'(\hat{\zeta}) = H(\hat{\zeta}) - \frac{H^2(\hat{\zeta})}{2}. \tag{83}$$

This nonlinear autonomous system has the critical points $(0,0)$ and $(2,0)$, which have different characteristics.

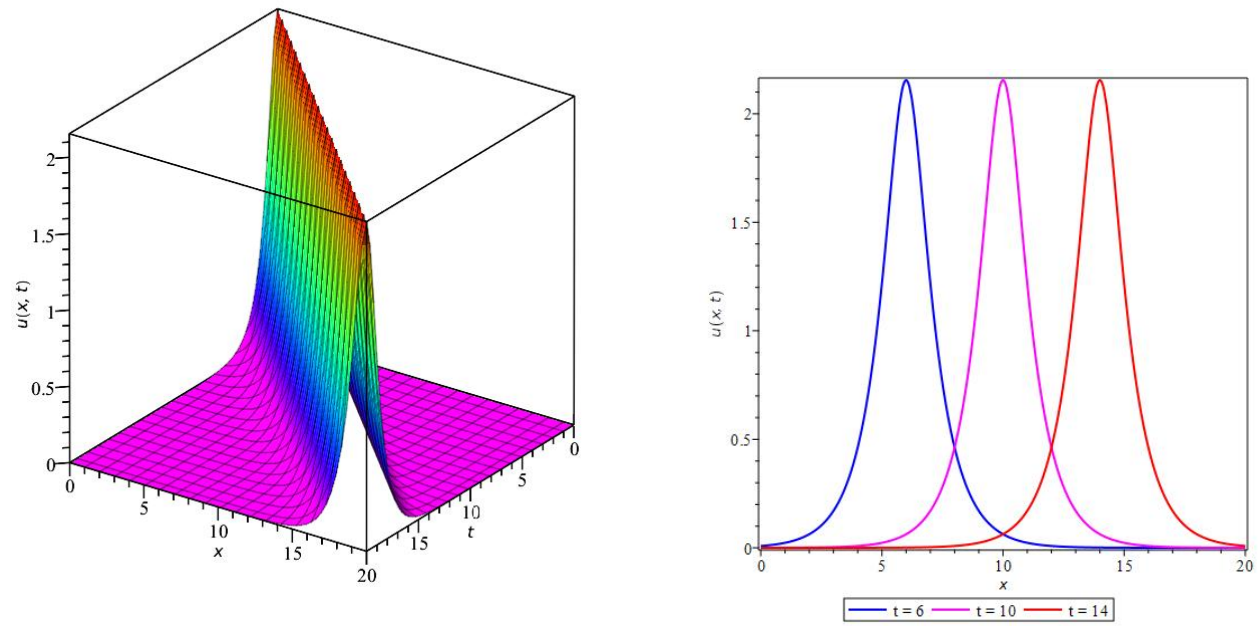


Figure 4: Representation of the bright-soliton solution in Eq. (82) for $\alpha = 1$ and $n = 3$.

At the critical point $(0, 0)$, the Jacobian matrix of the autonomous system is

$$\mathbf{J}(H, G) = \begin{pmatrix} 0 & 1 \\ 1 - H & 0 \end{pmatrix}. \tag{84}$$

Evaluating the Jacobian matrix at $(0, 0)$ gives

$$\mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{85}$$

The characteristic equation of the linearised system is

$$\lambda^2 - 1 = 0. \tag{86}$$

Thus, $\lambda_1 = -1$ and $\lambda_2 = 1$. The eigenvalues of the Jacobian matrix are real, distinct, and of opposite signs; therefore, the critical point $(0, 0)$ is an unstable saddle point, as shown in Figure 5.

At the critical point $(2, 0)$, introduce the shifted variables

$$h = H - 2, \qquad g = G. \tag{87}$$

The original autonomous system is then transformed into

$$h'(\hat{\zeta}) = g(\hat{\zeta}), \quad g'(\hat{\zeta}) = -h(\hat{\zeta}) - \frac{h^2(\hat{\zeta})}{2}. \tag{88}$$

The Jacobian matrix of the transformed system at $(h, g) = (0, 0)$ is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{89}$$

The characteristic equation of the linearised system is

$$\lambda^2 + 1 = 0. \tag{90}$$

Thus, $\lambda_{1,2} = \pm\iota$. The eigenvalues of the Jacobian matrix are purely imaginary. Because purely imaginary eigenvalues alone do not determine the stability of a nonlinear system, the linearisation is inconclusive. However, the system admits the first integral

$$G^2 - H^2 + \frac{H^3}{3} = C. \tag{91}$$

The level curves of this first integral in a neighbourhood of $(2, 0)$ are closed orbits. Therefore, the critical point $(2, 0)$ is a nonlinear centre; it is stable but not asymptotically stable, as shown in Figure 5.

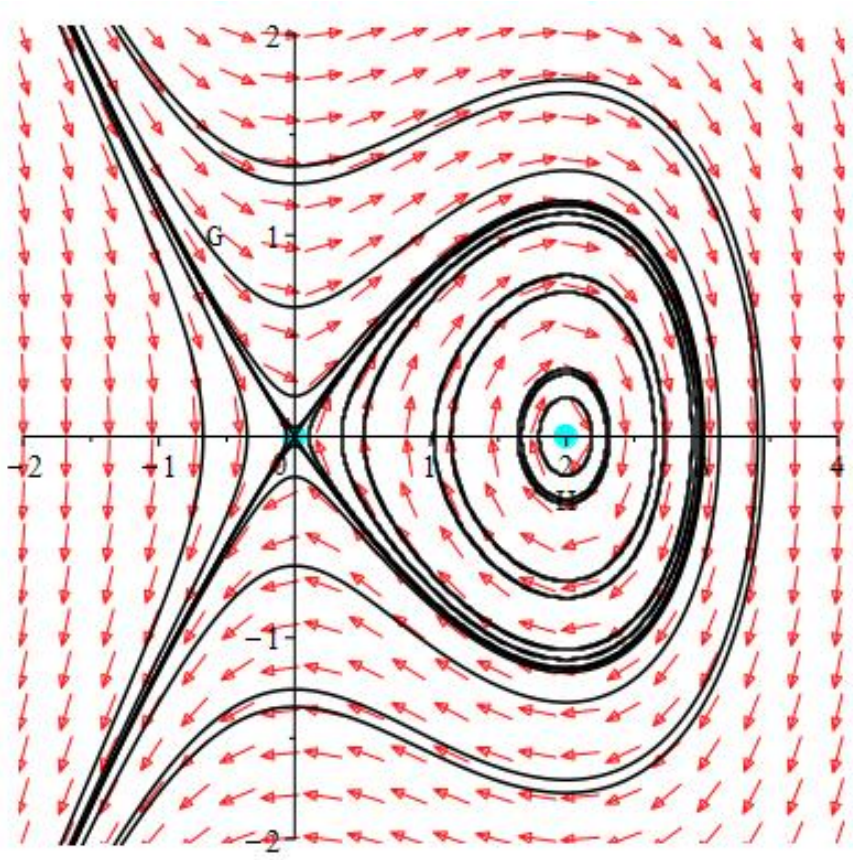


Figure 5: Phase portrait of the nonlinear autonomous system in Eq. (83), showing the saddle point $(0, 0)$ and the centre point $(2, 0)$.

4.2. *Bifurcation analysis corresponding to Eq. (42)*

Eq. (42) can be expressed as the following nonlinear autonomous system:

$$H'(\hat{\zeta}) = G(\hat{\zeta}), \qquad G'(\hat{\zeta}) = H(\hat{\zeta}) - \frac{H^3(\hat{\zeta})}{3}. \tag{92}$$

This nonlinear autonomous system has three critical points: $(0, 0)$, $(\sqrt{3}, 0)$, and $(-\sqrt{3}, 0)$, with different characteristics.

At the critical point $(0, 0)$, the Jacobian matrix of the autonomous system is

$$\mathbf{J}(H, G) = \begin{pmatrix} 0 & 1 \\ 1 - H^2 & 0 \end{pmatrix}. \tag{93}$$

Evaluating the Jacobian matrix at $(0, 0)$ gives

$$\mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{94}$$

The characteristic equation of the linearised system is

$$\lambda^2 - 1 = 0. \tag{95}$$

Thus, $\lambda_1 = -1$ and $\lambda_2 = 1$. The eigenvalues of the Jacobian matrix are real, distinct, and of opposite signs; therefore, the critical point $(0, 0)$ is an unstable saddle point, as shown in Figure 6.

At the critical point $(\sqrt{3}, 0)$, introduce the shifted variables

$$h = H - \sqrt{3}, \qquad g = G. \tag{96}$$

The original autonomous system is then transformed into

$$h'(\hat{\zeta}) = g(\hat{\zeta}), \quad g'(\hat{\zeta}) = -2h(\hat{\zeta}) - \sqrt{3}h^2(\hat{\zeta}) - \frac{h^3(\hat{\zeta})}{3}. \tag{97}$$

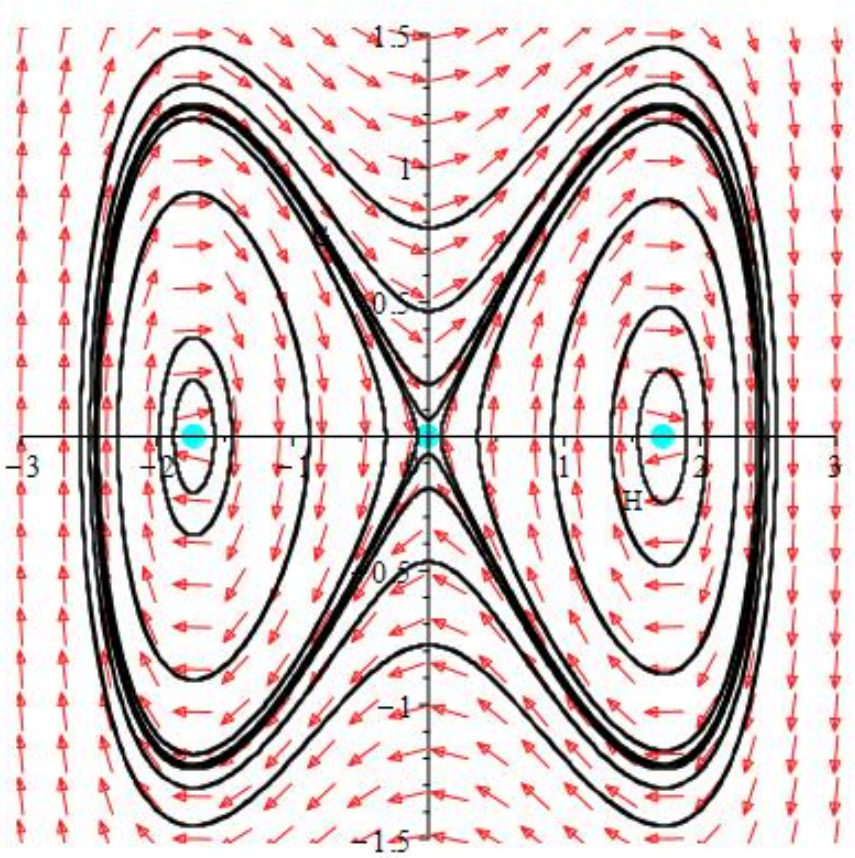


Figure 6: Phase portrait of the nonlinear autonomous system in Eq. (92), showing the saddle point $(0, 0)$ and the centre points $(\pm\sqrt{3}, 0)$.

The Jacobian matrix of the transformed system at $(h, g) = (0, 0)$ is

$$\mathbf{J}(0,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}. \tag{98}$$

The characteristic equation of the linearised system is

$$\lambda^2 + 2 = 0. \tag{99}$$

Thus, $\lambda_{1,2} = \pm\iota\sqrt{2}$. The eigenvalues of the Jacobian matrix are purely imaginary. Because purely imaginary eigenvalues alone do not determine the stability of a nonlinear system, the linearisation is inconclusive. However, the system admits the first integral

$$G^2 - H^2 + \frac{H^4}{6} = C. \tag{100}$$

The level curves of this first integral in a neighbourhood of $(\sqrt{3}, 0)$ are closed orbits. Therefore, the critical point $(\sqrt{3}, 0)$ is a nonlinear centre; it is stable but not asymptotically stable, as shown in Figure 6.

At the critical point $(-\sqrt{3}, 0)$, introduce the shifted variables

$$h = H + \sqrt{3}, \qquad g = G. \tag{101}$$

The original autonomous system is then transformed into

$$h'(\hat{\zeta}) = g(\hat{\zeta}), \quad g'(\hat{\zeta}) = -2h(\hat{\zeta}) + \sqrt{3}h^2(\hat{\zeta}) - \frac{h^3(\hat{\zeta})}{3}. \tag{102}$$

The Jacobian matrix of the transformed system at $(h, g) = (0, 0)$ is

$$\mathbf{J}(0,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}. \tag{103}$$

The characteristic equation of the linearised system is

$$\lambda^2 + 2 = 0. \tag{104}$$

Thus, $\lambda_{1,2} = \pm\iota\sqrt{2}$. The eigenvalues of the Jacobian matrix are purely imaginary. Because purely imaginary eigenvalues alone do not determine the stability of a nonlinear system, the linearisation is inconclusive. However, the system admits the first integral

$$G^2 - H^2 + \frac{H^4}{6} = C. \tag{105}$$

The level curves of this first integral in a neighbourhood of $(-\sqrt{3}, 0)$ are closed orbits. Therefore, the critical point $(-\sqrt{3}, 0)$ is a nonlinear centre; it is stable but not asymptotically stable, as shown in Figure 6.

5. Conclusion

In this work, Lie symmetry analysis was applied to three forms of time-space conformable fractional BBM equations. The corresponding symmetry generators and vector fields were derived, demonstrating the effectiveness of the Lie group approach for conformable fractional partial differential equations. Using these symmetries, the governing equations were reduced to ODEs, from which exact trigonometric and hyperbolic solutions were obtained using the (G'/G)-expansion method and suitable ansatz techniques. Bifurcation analysis and phase portraits were also used to investigate the qualitative behaviour of two reduced systems, providing insight into the nature and stability of their solutions. The results show that combining Lie symmetry analysis with analytical solution methods provides an effective framework for studying nonlinear conformable fractional BBM-type equations.

Data availability

No new data were generated or analysed in this study. Data sharing is not applicable to this article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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