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A Numerical Method using Collocation Approach for the Solution of Volterra-Fredholm Integro-Differential Equations

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Abstract

This paper consider collocation approach for the numerical solution of Volterra-Fredholm Integro-differential equation using collocation method. We transformed the problem into a system of linear algebraic equations and matrix inversion is adopted to solve the algebraic equations. We substituted the solution algebraic equations into the approximate equation to obtain the numerical result. Some numerical problems are solved to demonstrate the efficiency and consistency of the method.

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1. Introduction

Recently, research has focused on the most effective way to solve integral and integro-differential equations where the unknown function appears under the integral sign. Integro-differential equations are a common component of mathematical descriptions of physical phenomena. These equations can be found in a wide variety of contexts including fluid dynamics, biological models, and chemical kinetics. [1]

Many different approaches have been adopted to investigate the solution of integro-differential equations such as Adomian decompositions method by [2–4], Collocation method by [5–8], Hybrid linear multistep method [9],

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Chebyshev-Gelerkin method [10], Bernoulli matrix method [11], Differential transform method[12], Pseudospectral method [13], Bernstein polynomials method [14]. The Mellin transform approach [15], Homotopy Perturbation [16] and Perturbed method [17]. [18] developed a numerical method for solving the second kind of linear Volterra-Fredholm integro-differential equations using Bernestein polynomial. By using this method, the integro-differential equations are converted into an algebraic equation system and some numerical examples are offered. [19] proposed solution of Fractional Integro-differential Equations of Fredholm type using Legendre Galerkin method. The concept of Legendre Galerkin Method was implemented on some examples of fractional integro- differential equations of Fredholm type to illustrate the practicability of the method.

A numerical methodology based on quartic weighted polynomials for finding the solution of Fractional Integro-Differential Equations (FIDEs) was presented by [20]. The suggested method involved the application of the homotopy perturbation method and the initial approximation as the constructed orthogonal polynomials. [21] presented Akbari-Ganji's Method (AGM) to solve Volterra Integro-Differential Difference Equations (VIDDE) using Legendre polynomials as basis functions. A trial solution function of unknown constants that conform with the differential equations together with the initial conditions were assumed and substituted into the equations under consideration. Collocation approach for the computational solution of Fredholm-Volterra Fractional order of integro- differential equations was presented by [22]. They obtained the linear integral form of the problem and transformed it into a system of linear algebraic equations using standard collocation points. [23] considered first order Volterra Integro-differential equations using standard collocation method. An assumed approximate solution in terms of the constructed polynomial was substituted into the class of integro-differential equation considered. The resulted equation was collocated at appropriate points within the interval of consideration [0,1] to obtain a system of algebraic linear equations.
In this paper, we consider Volterra-Fredholm Integro-differential equation of the form:

$$y^{'}(x) = f(x) + \int_0^x k_1(x,t)y(t)dt + \int_0^1 k_2(x,t)y(t)dt \tag{1}$$

subject to initial condition

$$y(a) = 0 \tag{2}$$

$a \leq x \leq b$, $y(x)$ is the unkown function, $k_1(x,t)$ and $k_2(x,t)$ are the Volterra and Fredholm integral kanel function respectively. $f(x)$ is the known function

2. Basic Definitions

In this section,we provide certain definitions and fundamental ideas for the formulation of the specified problem.
Definition 1: Let $(a_n), n \geq 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_N x^N = \sum_{n=0}^{N} a_n x^n = \phi(x)\,\mathbf{A} \tag{3}$$

where $\phi(x) = [1 \quad x \quad x^2 \quad \cdots \quad x^N]$, $\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$
then $y(x,n) = x^n\mathbf{A}$, $n = 0(1)N$, $n \in Z^+$
Definition 2: The desired collocation points within an interval are determined using this method. i.e. [a,b] and is provided by

$$x_i = a + \frac{(b-a)i}{N}, i = 1, 2, 3, ...N \tag{4}$$

3. Methodology

In this section, combination of collocation method and power series approximation are employed for the numerical solution of volterra-fredholm integro-differential equations.
Let the solution to (1) and (3) be approximated by

$$y(x) = \phi(x)\,\mathbf{A} \tag{5}$$

$\phi(x)$ is an interpolating polynomial and **A** are parameters to be determined,

$$\phi(x) = \begin{bmatrix} \phi_0(x) & \phi_1(x) & \phi_2(x) & \cdots \phi_N(x) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_N \end{bmatrix}^T$$

substituting (3) into (1) gives

$$\phi'(x)\mathbf{A} = f(x) + \int_0^x k_1(x,t)\phi(t)\mathbf{A}dt + \int_0^1 k_2(x,t)\phi(t)\mathbf{A}dt \tag{6}$$

collecting the like terms

$$\left(\phi'(x) - \int_0^x k_1(x,t)\phi(t)\,dt + \int_0^1 k_2(x,t)\phi(t)\,dt\right)\mathbf{A} = f(x) \tag{7}$$

Equation (7) can be writing in this form

$$U(x)\mathbf{A} = f(x) \tag{8}$$

where

$$U(x) = \left(\phi'(x) - \int_0^x k_1(x,t)\phi(t)\,dt + \int_0^1 k_2(x,t)\phi(t)\,dt\right)_{1\times[N+1]}$$

Collocating (8) using the standard collocation points

$$x_i = a + \frac{b-a}{N}i$$

$$U(x_i)\mathbf{A} = f(x_i) \tag{9}$$

where

$$U(x_i) = \begin{bmatrix} U_0(x_0) & U_1(x_0) & U_2(x_0) & \cdots & U_N(x_0) \\ U_0(x_1) & U_1(x_1) & U_2(x_1) & \cdots & U_N(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_0(x_N) & U_1(x_N) & U_2(x_N) & \cdots & U_N(x_N) \end{bmatrix}, \; f(x_i) = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

Using the initial conditions

$$y(a) = 0 \tag{10}$$

hence, (10) becomes

$$\phi(a) = 0 \tag{11}$$

Substituting (11) into equation (9) gives

$$U^*(x_i)\mathbf{A} = f^*(x_i) \tag{12}$$

The values of the unknown are solved using matrix inversion. Substituting the values of a_i, $i = 0, 1, 2, ..., N$ obtained into the approximate solution gives the numerical solution

$$y(x) = \phi(x_i)U^{*-1}(x_i)f^*(x_i) \tag{13}$$

4. Numerical Examples

In this section, two numerical examples with initial conditions are presented to confirm the efficiency and accuracy of the method. Let $y_n(x)$ and $y(x)$ be the approximate and exact solution respectively. $\text{Error}_N = |y_n(x) - y(x)|$
Example 1: [18] Considering volterra-fredholm integro-differential equation

$$y'(x) = f(x) - \int_0^x x^2 t y(t)dt - \int_0^1 x(x-t)y(t)dt$$

subject to initial conditions

$$y(0) = 1,\ 0 \le x \le 1$$

where

$$f(x) = -sin(x) + x^2 cos(x) + x^3 sin(x) - x^2 + x^2 sin(1) - xcos(1) - xsin(1) + x$$

Exact solution $y(x) = cox(x)$
Solution 1
We solve this problem at $N = 5, 6$ *and* 10 but we use $N = 4$ for demonstration
Using approximate solution (3) on example 1 gives

$$\phi'(x)\mathbf{A} = f(x) - \int_0^x x^2 t\, \phi(t)\mathbf{A}dt - \int_0^1 x(x-t)\, \phi(t)\mathbf{A}dt$$

$$\left[\phi'(x) + \int_0^x x^2 t\, \phi(t)dt + \int_0^1 x(x-t)\, \phi(t)\, dt\right]\mathbf{A} = f(x) \tag{14}$$

Equation (14) gives

$$U(x)\mathbf{A} = f(x) \tag{15}$$

where

$$U(x) = \phi'(x) + \int_0^x x^2 t\, \phi(t)dt + \int_0^1 x(x-t)\, \phi(t)\, dt$$

collocating at $x_5 = \begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix}$ and substituting the initial conditions gives

$$U(x_i)^* \mathbf{A} = f(x_i)^* \tag{16}$$

where

$$U(x_i) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{37}{625} & \frac{5959}{6250} & \frac{17032}{46875} & \frac{140629}{1562500} & \frac{7813}{1171875} & -\frac{1140619}{82031250} \\ -\frac{17}{625} & \frac{2969}{3125} & \frac{70721}{93750} & \frac{172003}{390625} & \frac{86501}{390625} & \frac{571648}{5859375} \\ \frac{78}{625} & \frac{6287}{6250} & \frac{36927}{31250} & \frac{1649373}{1562500} & \frac{327656}{390625} & \frac{17054991}{27343750} \\ \frac{278}{625} & \frac{3633}{3125} & \frac{78697}{46875} & \frac{766384}{390625} & \frac{2426518}{1171875} & \frac{84473932}{41015625} \\ 1 & \frac{3}{2} & \frac{7}{3} & \frac{13}{4} & \frac{21}{5} & \frac{31}{6} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f(x_i) = \begin{bmatrix} 0 & -0.2405731317 & -0.3951997680 & -0.4316932976 & -0.3110546806 & 0 & 1 \end{bmatrix}^T$$

We now solve for the unknown values **A** (16) making use of matrix inversion results in;

$$y_5 = \begin{pmatrix} 1.000000014500000 - 0.1100900000e - 5x - 0.499665406900000x^2 \\ -0.1934020400000e - 2x^3 + 0.45850805300000e - 1x^4 - 0.3948219000000e - 2x^5 \end{pmatrix}$$

Applying the same procedure for $N = 6$ *and* 10 gives

$$y_6 = \begin{pmatrix} 0.999999999900000 - 1.094862422900000 \times 10^{-8}x - 0.499989658700000x^2 \\ -0.79348700000e - 4x^3 + 0.41903091000000e \\ -1x^4 - 0.323985800000e - 3x^5 - 0.1207684000000e - 2x^6 \end{pmatrix}$$

Table 1. Exact and approximate values, Example 1

x	Exact	$N = 5$	$N = 6$	$N = 10$
0.2	0.980066577841242	0.980069803739200	0.980066640551043	0.980066578523643
0.4	0.921060994002885	0.921062682583520	0.921061026653094	0.921060999303860
0.6	0.825335614909678	0.825337309927040	0.825335678632713	0.825335614259620
0.8	0.696706709347165	0.696709792368160	0.696706738330861	0.696706709330861
1.0	0.540302305868140	0.540302072600000	0.540302402751376	0.540302305851376

Table 2. Absolute Error for Example 1

x	$error_5$	$error_6$	$error_{10}$
0.2	0.000003225897958	6.2709801000000001e-8	7.236430000000000e-10
0.4	0.000001688580635	3.265020900000000e-8	5.300975000000000e-9
0.6	0.000001695017362	6.372303500000000e-8	1.646500580000000e-9
0.8	0.000003083020995	2.898369600000000e-8	1.8604468190000000e-9
1.0	2.332681400000000e-7	9.688323600000000e-8	1.0605826775000000e-9

$$y_{10} = \begin{pmatrix} 1.000000000000000 - 5.144329406903130 \times 10^{-12} x - 0.499999944586307 x^2 \\ -9.462237358093260 \times 10^{-7} x^3 + 0.41672945022583e - 1x^4 - 0.18835067749e - 4x^5 \\ -0.1343965530396e - 2x^6 - 0.78201293945e - 4x^7 \\ +0.87261199951e - 4x^8 - 0.31232833862e - 4x^9 + 0.4619359970e - 5x^{10} \end{pmatrix}$$

Example 2 [18] Considering fractional integro-differential equation

$$y'(x) = f(x) + \int_0^x y(t)dt + \int_0^1 y(t)dt$$

subject to initial conditions

$$y(0) = 0, \ 0 \le x \le 1$$

where

$$f(x) = 2e^x - 2$$

Exact solution $y(x) = xe^x$

Solution 2

Using approximate solution (3) on example 2 gives

$$\phi'(x)\,\mathbf{A} = f(x) + \int_0^x \phi(t)\,\mathbf{A}dt + \int_0^1 \phi(t)\,\mathbf{A}dt$$

$$\left[\phi'(x) - \int_0^x \phi(t)dt - \int_0^1 \phi(t)\,dt\right]\mathbf{A} = f(x) \tag{17}$$

Equation (17) gives

$$U(x)\mathbf{A} = f(x) \tag{18}$$

where

$$U(x) = \phi'(x) - \int_0^x \phi(t)dt - \int_0^1 \phi(t)\,dt$$

collocating at $x_4 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}$ and substituting the initial conditions gives

$$U(x_i)^*\mathbf{A} = f(x_i)^* \tag{19}$$

Table 3. Exact and approximate values, Example 2

x	Exact	$N = 4$	$N = 6$	$N = 10$
0.2	0.244280551632034	0.245107750566580	0.244280474366985	0.244280551735439
0.4	0.596729879056508	0.597952145575241	0.596729787759204	0.596729879059536
0.6	1.093271280234310	1.094251609446400	1.093271159740840	1.093271280740840
0.8	1.780432742793980	1.781874552875250	1.780432577181920	1.7804327427932846
1.0	2.718281828459050	2.720832262494860	2.718281685733800	2.7182818284583942

Table 4. Absolute Error for Example 2

x	error_4	error_6	error_{10}
0.2	0.000827198934546	7.726504900000000e-8	1.010340500000000e-10
0.4	0.001222266518733	9.129730400000000e-8	1.090302800000000e-10
0.6	0.000980329212090	1.204934700000000e-7	2.6958374929600000e-9
0.8	0.001441810081270	1.656120600000000e-7	1.0116729458000000e-9
1.0	0.002550434035810	1.427252500000000e-7	3.5968446284000000e-9

where

$$U(x_i) = \begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} \\ -\frac{5}{4} & \frac{15}{32} & \frac{31}{192} & -\frac{65}{1024} & -\frac{141}{1024} \\ -\frac{3}{2} & \frac{3}{8} & \frac{5}{8} & \frac{31}{64} & \frac{47}{160} \\ -\frac{7}{4} & \frac{7}{32} & \frac{197}{192} & \frac{1391}{1024} & \frac{7373}{5120} \\ -2 & 0 & \frac{4}{3} & \frac{5}{2} & \frac{18}{5} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f(x_i) = \begin{bmatrix} 0 & 0.568050833375 & 1.29744254140 & 2.23400003322534 & 3.4365636569181 & 0 \end{bmatrix}^T$$

We now solve for the unknown values **A** (19) making use of matrix inversion results in;

$$y_4 = \begin{pmatrix} -0.7110336836e-5 + 1.000614432260340x \\ +1.039187111971780x^2 + 0.364817101049653x^3 + 0.316220727549933x^4 \end{pmatrix}$$

Applying the same procedure for $N = 7$ *and* 10 gives

$$y_6 = \begin{pmatrix} -1.728131526768100e-14 + 0.999999894930511x \\ +0.999982466411893x^2 + 0.500181985087693x^3 + 0.165893773548305x^4 \\ +0.43341163545847e-1x^5 + 0.6405057851225e-2x^6 + 0.2477344358340e-2x^7 \end{pmatrix}$$

$$y_{10} = \begin{pmatrix} -7.646335821454020e-15 + 0.999999999883585x + 0.999999657273293x^2 \\ +0.500003099441528x^3 + 0.166658401489258x^4 + 0.41748046875000e-1x^5 \\ +0.008087158203125x^6 + 0.001541137695313x^7 + 0.000213623046875x^8 \\ +0.000007629394531x^9 - 0.000005722045898x^{10} \end{pmatrix}$$

5. Conclusion

Volterra-Fredholm integro-differential equations are complex to solve analytically. Hence, approximate solution is needed. This paper considered collocation method for the solution of volterra-fredholm integro-differential equations. The method is consistent, efficient and easy to compute. The results of example 1 as shown in the table 1and 2 shows the approximate solutions converges to the exact solution when the values of N increases. In example 2, The approximate solution at $N = 4$ gives $y_4(x) = -0.7110336836e-5 + 1.000614432260340x + 1.039187111971780x^2 + 0.364817101049653x^3 + 0.316220727549933x^4$ and solving for N=6 and 10, we obtained Table 3 which display the results obtained at $x = 0.2$ *to* 1.0 for various values of N and the exact solution. Error of example 2 as shown in Table 4 indicates that as the values of N increases, the error becomes smaller. All computations are done with the aid of Maple 18.

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