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Sharp fractional integral inequalities for generalized logarithmic Godunova-Levin functions via Riemann-Liouville operators

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Abstract

We introduce a new class of generalized logarithmic Godunova-Levin interval-valued mappings and establish weighted Hermite-Hadamard and Jensen-type inequalities via Riemann-Liouville fractional integrals. Results for real-valued mappings follow as special cases, along with recovery of several known inequalities. Illustrative examples with tabular numerical validation confirm the theoretical findings.

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1. Introduction

A fundamental result in convex function theory is the Hermite-Hadamard integral inequality, a well-established theorem in the mathematical literature that serves as a key tool in convex analysis and the study of inequalities.

Theorem 1.1 ([1]). Suppose $\mathfrak{F} : \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$ is a convex function on an interval $\mathcal{I}$, and let $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{I}$ satisfy $\mathfrak{v}_1 < \mathfrak{v}_2$. Then the subsequent double inequality is valid:

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \leq \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k} \leq \frac{\mathfrak{F}(\mathfrak{v}_1) + \mathfrak{F}(\mathfrak{v}_2)}{2}. \tag{1}$$

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Inequality Eq. (1) has been extensively generalized and refined in various directions. One notable extension arises from the class of log-convex functions. A function $\mathfrak{F} : \mathcal{I} \to (0, \infty)$ is said to be log-convex if $\ln \mathfrak{F}$ is convex on $\mathcal{I}$. In this setting, the following refinement of the Hermite-Hadamard inequality holds (see [2], p. 197, (5.3)):

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \le \exp\left[\frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k}\right] \le \sqrt{\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)}. \tag{2}$$

In 2007, S. Varošanec [3] introduced the notion of h-convex functions, providing a unifying generalization of classical convexity. Within this framework, numerous Hermite-Hadamard-type inequalities have been derived. In particular, Noor *et al.* [4] introduced the class of log-h-convex functions and established the following refinement of inequality Eq. (2).

Theorem 1.2 ([4])**.** Let $\mathfrak{F} : \mathcal{I} \to (0, \infty)$ be a log-h-convex function with $h\left(\frac{1}{2}\right) \neq 0$. Then

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \le \exp\left[\frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k}\right] \le [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 h(t)\, dt}. \tag{3}$$

Motivated by the increasing significance of interval-valued analysis, recent studies have focused on extending these types of inequalities to functions that take values in interval form. In particular, Afzal *et al.* [5] investigated such inequalities for numerous classes of mappings, including Godunova-Levin-type functions, and established several extensions in both convex and harmonically convex settings.

Theorem 1.3 ([5])**.** Suppose $h : (0, 1) \to \mathcal{R}^+$ is a function for which $h\left(\frac{1}{2}\right) \neq 0$. Let $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}^+$ satisfies $\mathfrak{F} \in SGX(\text{cr-}h, [\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^+)$. Then the following result holds:

$$\frac{h\left(\frac{1}{2}\right)}{2}\, \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \preceq_{\text{cr}} \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k} \preceq_{\text{cr}} [\mathfrak{F}(\mathfrak{v}_1) + \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \frac{dt}{h(t)}. \tag{4}$$

The theory of convex functions and their associated integral inequalities has remained a cornerstone of mathematical analysis, with the Hermite-Hadamard (H-H) inequality serving as one of its most extensively studied pillars. In recent years, this classical framework has been substantially generalized to interval-valued and fuzzy-interval-valued settings, enabling the treatment of uncertainty within convex analysis. According to Ref. [6], the authors investigated new refinements of convex inequalities using values over intervals. Building on this direction, Abdelfattah *et al.* [7] introduced new fractional developments of Hermite-Hadamard-type inequalities, extending classical bounds through generalized fractional operators. Similarly, in Ref. [8], the authors derived fractional convex inequalities in n-polynomial type convex generalized mappings. Further contributions to two-dimensional and coordinated convexity have enriched this literature. Khan *et al.* [9] developed some related convex integral inequalities and product form in coordinated convex type mappings, while Ali *et al.* [10] used the q-Jackson quantum integral operator and developed some relevant types of inequalities. Zhao *et al.* [11] extended these ideas to coordinated h-convex interval-valued functions, and Ahmad *et al.* [12] investigated a specific new class of convex Hermite-Hadamard-type integral inequalities with applications in certain directions.

Parallel to these developments, the theory of topological and soft-set operators has advanced considerably. Alghamdi *et al.* [13] introduced novel operators within primal topological spaces, and Tariq *et al.* [14] investigated some new convex integral inequalities of Hermite-Hadamard-type involving Mittag-Leffler functions using a fractional approach. Ameen and Alqahtani [15] examined Baire category soft sets and their symmetric local properties, whereas Srivastava *et al.* [16] derived Hermite-Hadamard-type inequalities for interval-valued preinvex functions through fractional integral operators. Lai *et al.* [17] worked on preinvex type mappings that naturally generalize classical convex functions and developed various new product forms of related inequalities. Topological separation and closure structures have also been actively investigated. Alqahtani and Latif [18] discussed some new properties of operators defined over function spaces and topology, and Adrees *et al.* [19] worked on three related types of convex inequalities in a stochastic setting, defined over random variables and a probabilistic approach. In a related applied context, in [20], the authors proposed approximate solutions of some nonlinear evolution equations arising in physical mathematics, while Tan *et al.* [21] investigated Hermite-Hadamard inequalities for $\bar{h}$-preinvex interval-valued functions via fractional integrals. Additional refinements to fractional inequalities and their applications include Abdelfattah *et al.* [22], who derived fractional integral inequalities associated with special means, and Liu *et al.* [23], who established new Hermite-Hadamard and Jensen inequalities in fuzzy setting. Al-Omari and Alqahtani [24] worked on some closure properties of operators in primal ordered structure, and Afzal *et al.* [25] investigated such types of inequalities defined in stochastic probabilistic setting. Geometric function theory and soft topology have likewise contributed to this broad inequality-analysis landscape. Bello *et al.* [26] examined some new subordination formulas associated with a quantum derivative operator, while in [27] the authors worked on soft set theory operator type problems. Zhang *et al.* [28] first introduced a class of Godunova-Levin function whose mapping range is defined over intervals and then utilized it to develop some related types of inequalities, and Adrees *et al.* [29] developed some properties associated with probabilistic convex mappings defined over probability space and developed new Hermite-Hadamard and relevant inequalities. Further soft-topological advances appear in

El-Latif and Alqahtani [30], who classified novel categories of soft-type continuous order operators, while Akram *et al.* [31] derived Hermite-Hadamard inequalities for Godunova-Levin mapping defined over Jackson quantum integral operator of fractional order. In a similar vein, Khan *et al.* [32] used the idea of superquadratic maps defined over fuzzy domain and their associated integral inequalities, and Afzal *et al.* [33] established boundedness and Sobolev-type estimates for the exponentially damped Riesz potential, applying them to elliptic PDE regularity theory.

Stochastic-process-based inequalities have gained particular attention as well. Abbas *et al.* [34] derived Jensen, Ostrowski, and Hermite-Hadamard-type inequalities for h-convex stochastic processes using center-radius order relations, and Shah *et al.* [35] obtained fractional Hermite-Hadamard-Mercer inequalities for interval-valued convex stochastic processes with applications in entropy and information theory. Zhou and Du [36] investigated reverse Minkowski and reverse Hölder type fractional inclusions arising from interval-valued mappings, while Afzal *et al.* [37] proposed novel estimates for harmonically convex mappings via center-radius order relations. In 2025, Sahoo *et al.* [38] introduced new fractional Hermite-Hadamard inequalities for harmonically convex functions by employing generalized proportional fractional integral operators. Preinvexity and Godunova-Levin convexity remain active research threads. In [39], the authors presented some relevant types of inequalities using generalized preinvex functions, while Ali *et al.* [40, 41] derived Hermite-Hadamard and their product form using fractional Riemann-Liouville type operators in preinvex functions, and Farid *et al.* [42] established Ostrowski-type inequalities using fractional integrals. Finally, recent contributions continue to push these frameworks toward superquadratic and stability-oriented directions. Ahmad *et al.* [43] presented a new form of convex Hadamard type inequalities using superquadratic functions, and Afzal *et al.* [44] worked on two-dimensional convex functions and discussed various new properties and various product type inequalities, along with some stability problems for future studies.

Beyond interval-valued convexity, closely related geometric and fractional-order inequality frameworks have also informed this study. Liu *et al.* [45] established new inequalities for cr-log-h-convex functions, extending classical logarithmic convexity notions to the interval setting. In [46], the authors derived a discrete analogue of the generalized type of Minkowski inequality, offering combinatorial insight into classical convex-geometric bounds, while Akman *et al.* [47], using these results, worked on some practical applications defined in form of elliptic PDEs. Almutairi and Kılıçman [48] established integral inequalities for mappings defined over invex set, and Saeed and Treanţă [49] worked on some variational inequalities and some applications related to them. Cheng *et al.* [50] developed fractional convex inequalities of that type using q-Jackson integral operator, and Kashuri *et al.* [51] formulated new analogues of Hermite-Hadamard-type inequalities for subadditive functions via tempered fractional integrals.

The novelty and significance of this study lie in the introduction of a generalized logarithmic Godunova-Levin function that unifies several existing works. Unlike prior contributions that employ classical integral operators on related convex classes [2, 4], we adopt Riemann-Liouville fractional operators to derive more general and refined inequalities. The results obtained are further validated through examples and remarks, demonstrating recovery of known results under appropriate parameter and fractional-order choices.

The article is organized as follows. In Section 1, we review the relevant literature and provide the motivation behind the problems studied. Section 2 recalls some necessary notions, including definitions, operators, and concepts of interval calculus. Section 3 contains our major outcomes, in which we establish several inequalities, such as weighted Hermite-Hadamard and Jensen-type inequalities, for functions with certain convexity properties. Finally, in Section 4, we conclude the article with a discussion of the results and propose directions for future research.

2. Preliminaries

In this section, we recall some concepts related to interval analysis, classical convex mappings involving logarithmic concepts, and their interval-valued forms.

2.1. Operations on intervals

Let $\mathcal{R}_{\mathcal{I}}$ denote the family of all nonempty closed intervals in $\mathcal{R}$. An interval $\mathfrak{v}_1 = [\underline{\mathfrak{v}_1}, \overline{\mathfrak{v}_1}]$ is said to be *positive* if $\underline{\mathfrak{v}_1} > 0$. The collection of all such positive intervals is denoted by $\mathcal{R}_{\mathcal{I}}^{+}$, while $\mathcal{R}^{+}$ represents the set of all positive real numbers.

For any scalar $\kappa \in \mathcal{R}$, the Minkowski addition and scalar multiplication are defined by

$$\mathfrak{v}_1 + \mathfrak{v}_2 = [\underline{\mathfrak{v}_1} + \underline{\mathfrak{v}_2},\ \overline{\mathfrak{v}_1} + \overline{\mathfrak{v}_2}],$$

and

$$\kappa\mathfrak{v}_1 = \begin{cases} [\kappa\underline{\mathfrak{v}_1},\ \kappa\overline{\mathfrak{v}_1}], & \kappa > 0, \\ \{0\}, & \kappa = 0, \\ [\kappa\overline{\mathfrak{v}_1},\ \kappa\underline{\mathfrak{v}_1}], & \kappa < 0. \end{cases}$$

For $\mathfrak{v}_1 = [\underline{\mathfrak{v}_1}, \overline{\mathfrak{v}_1}] \in \mathcal{R}_{\mathcal{I}}$, its *center* and *radius* are defined, respectively, by

$$\mathfrak{v}_{1c} = \frac{\overline{\mathfrak{v}_1} + \underline{\mathfrak{v}_1}}{2}, \qquad \mathfrak{v}_{1r} = \frac{\overline{\mathfrak{v}_1} - \underline{\mathfrak{v}_1}}{2}.$$

Thus, $\mathfrak{v}_1$ admits the center-radius representation

$$\mathfrak{v}_1 = \langle \mathfrak{v}_{1\mathfrak{c}}, \mathfrak{v}_{1\mathrm{r}} \rangle.$$

Definition 2.1 ([5]). Let $\mathfrak{v}_1 = \langle \mathfrak{v}_{1\mathfrak{c}}, \mathfrak{v}_{1\mathrm{r}} \rangle$ and $\mathfrak{v}_2 = \langle \mathfrak{v}_{2\mathfrak{c}}, \mathfrak{v}_{2\mathrm{r}} \rangle$ be two intervals in $\mathcal{R}_{\mathcal{I}}$. The total interval order, commonly referred to as the cr-order, is defined as follows:

$$\mathfrak{v}_1 \preceq_{\mathrm{cr}} \mathfrak{v}_2 \Longleftrightarrow \begin{cases} \mathfrak{v}_{1\mathfrak{c}} < \mathfrak{v}_{2\mathfrak{c}}, & \text{if } \mathfrak{v}_{1\mathfrak{c}} \neq \mathfrak{v}_{2\mathfrak{c}}, \\ \mathfrak{v}_{1\mathrm{r}} \leq \mathfrak{v}_{2\mathrm{r}}, & \text{if } \mathfrak{v}_{1\mathfrak{c}} = \mathfrak{v}_{2\mathfrak{c}}. \end{cases}$$

Moreover, for any $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{R}_{\mathcal{I}}$, either $\mathfrak{v}_1 \preceq_{\mathrm{cr}} \mathfrak{v}_2$ or $\mathfrak{v}_2 \preceq_{\mathrm{cr}} \mathfrak{v}_1$ holds.

The idea of Riemann integration defined in terms of intervals was introduced in [5], and can be expressed as follows:

Theorem 2.1 ([5]). For $\mathfrak{F} = [\underline{\mathfrak{F}}, \overline{\mathfrak{F}}] : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}$, the function $\mathfrak{F}$ is Riemann integrable on $[\mathfrak{v}_1, \mathfrak{v}_2]$ iff $\underline{\mathfrak{F}}$ and $\overline{\mathfrak{F}}$ are both Riemann integrable on $[\mathfrak{v}_1, \mathfrak{v}_2]$. In this case,

$$\int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \mathfrak{F}(\mathfrak{d})\, d\mathfrak{d} = \left[\int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \underline{\mathfrak{F}}(\mathfrak{d})\, d\mathfrak{d}, \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \overline{\mathfrak{F}}(\mathfrak{d})\, d\mathfrak{d} \right].$$

We denote by $\mathcal{IR}([\mathfrak{v}_1, \mathfrak{v}_2])$ the collection of all interval-valued functions that are Riemann integrable on $[\mathfrak{v}_1, \mathfrak{v}_2]$.

Theorem 2.2 ([5]). Let $\mathfrak{F}, \mathcal{G} \in \mathcal{IR}([\mathfrak{v}_1, \mathfrak{v}_2])$ take values in $\mathcal{R}_{\mathcal{I}}^{+}$. If

$$\mathfrak{F}(\mathfrak{d}) \preceq_{\mathrm{cr}} \mathcal{G}(\mathfrak{d}), \qquad \forall \mathfrak{d} \in [\mathfrak{v}_1, \mathfrak{v}_2],$$

then

$$\int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \mathfrak{F}(\mathfrak{d})\, d\mathfrak{d} \preceq_{\mathrm{cr}} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \mathcal{G}(\mathfrak{d})\, d\mathfrak{d}.$$

That is, the cr-order is preserved under integration.

More details on interval analysis can be found in [5].

2.2. Fundamentals of convexity in real and interval settings

Definition 2.2 ([1]). A function $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}$ is said to be *convex* if

$$\mathfrak{F}(\iota\mathfrak{d} + (1-\iota)\kappa) \leq \iota\mathfrak{F}(\mathfrak{d}) + (1-\iota)\mathfrak{F}(\kappa),$$

for all $\mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2]$ and $\iota \in [0, 1]$.

Definition 2.3 ([1]). A function $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}$ is said to be of *Godunova-Levin-type* if

$$\mathfrak{F}(\iota\mathfrak{d} + (1-\iota)\kappa) \leq \frac{\mathfrak{F}(\mathfrak{d})}{\iota} + \frac{\mathfrak{F}(\kappa)}{1-\iota},$$

for all $\mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2]$ and $\iota \in (0, 1]$.

Definition 2.4 ([44]). Let $\mathfrak{F} = [\underline{\mathfrak{F}}, \overline{\mathfrak{F}}] : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}$. Then $\mathfrak{F}$ is *interval-valued convex* on $[\mathfrak{v}_1, \mathfrak{v}_2]$ iff $\underline{\mathfrak{F}}$ and $\overline{\mathfrak{F}}$ are convex.

$$\mathfrak{F}(\iota\mathfrak{d} + (1-\iota)\kappa) \preceq_{\mathrm{cr}} \iota\mathfrak{F}(\mathfrak{d}) + (1-\iota)\mathfrak{F}(\kappa),$$

for all $\mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2]$ and $\iota \in [0, 1]$.

Equivalently, both endpoint functions $\underline{\mathfrak{F}}$ and $\overline{\mathfrak{F}}$ are convex on $[\mathfrak{v}_1, \mathfrak{v}_2]$ and satisfy

$$\underline{\mathfrak{F}}(\mathfrak{d}) \leq \overline{\mathfrak{F}}(\mathfrak{d}), \qquad \forall \mathfrak{d} \in [\mathfrak{v}_1, \mathfrak{v}_2].$$

The class of all such interval-valued convex functions mapping into $\mathcal{R}_{\mathcal{I}}^{+}$ is denoted by

$$\mathcal{IVCX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^{+}).$$

Definition 2.5 ([37]). Let $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}$ be written in center-radius form as

$$\mathfrak{F} = \langle \mathfrak{F}_{\mathfrak{c}}, \mathfrak{F}_{\mathrm{r}} \rangle,$$

where

$$\mathfrak{F}_{\mathfrak{c}}(\mathfrak{d}) = \frac{\overline{\mathfrak{F}}(\mathfrak{d}) + \underline{\mathfrak{F}}(\mathfrak{d})}{2}, \qquad \mathfrak{F}_{\mathrm{r}}(\mathfrak{d}) = \frac{\overline{\mathfrak{F}}(\mathfrak{d}) - \underline{\mathfrak{F}}(\mathfrak{d})}{2}.$$

Then $\mathfrak{F}$ is known as cr-*convex* on $[\mathfrak{v}_1, \mathfrak{v}_2]$ if, for all $\mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2]$ and $\iota \in [0, 1]$,

$$\mathfrak{F}_{\mathfrak{c}}(\iota\mathfrak{d} + (1-\iota)\kappa) \leq \iota\mathfrak{F}_{\mathfrak{c}}(\mathfrak{d}) + (1-\iota)\mathfrak{F}_{\mathfrak{c}}(\kappa),$$

and

$$\mathfrak{F}_{\mathrm{r}}(\iota\mathfrak{d} + (1-\iota)\kappa) \leq \iota\mathfrak{F}_{\mathrm{r}}(\mathfrak{d}) + (1-\iota)\mathfrak{F}_{\mathrm{r}}(\kappa).$$

Remark 1. For any interval-valued function $\mathfrak{F}$, the following implications hold:

- If $\mathfrak{F}$ is cr-convex, then it is interval-valued convex.
- If $\mathfrak{F}$ is interval-valued convex, then both $\underline{\mathfrak{F}}$ and $\overline{\mathfrak{F}}$ are convex in the classical sense.

2.3. Log-convexity and associated function classes

Definition 2.6 ([4]). Let $h : [0,1] \to \mathcal{R}^+$ be a given function. A mapping $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}^+$ is said to be log-*h-convex* if

$$\mathfrak{F}(\iota\mathfrak{d} + (1-\iota)\kappa) \leq [\mathfrak{F}(\mathfrak{d})]^{h(\iota)}[\mathfrak{F}(\kappa)]^{h(1-\iota)},$$

for all $\mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2]$ and $\iota \in [0,1]$.

The function h is called *supermultiplicative* if

$$h(\vartheta\iota) \geq h(\vartheta)h(\iota), \qquad \forall \vartheta, \iota \in [0,1].$$

Definition 2.7 ([5]). For $\mathfrak{F} = [\underline{\mathfrak{F}}, \overline{\mathfrak{F}}] : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}^+$ and $h : [0,1] \to \mathcal{R}^+$, define $\mathfrak{F}$ as cr-log-*h-convex* if

$$\forall \mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2],\ \forall \iota \in [0,1] : \quad \mathfrak{F}(\iota\mathfrak{d} + (1-\iota)\kappa) \ \preceq_{\text{cr}} \ [\mathfrak{F}(\mathfrak{d})]^{h(\iota)}[\mathfrak{F}(\kappa)]^{h(1-\iota)},$$

where exponentiation and multiplication are componentwise.

3. Main results

This section is devoted to developing some new refinements of Hermite-Hadamard and Jensen-type inequalities by defining a new related class of logarithmic convex mappings via the Riemann-Liouville fractional operator.

Definition 3.1. Let $\mathfrak{F} = [\underline{\mathfrak{F}}, \overline{\mathfrak{F}}] : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}^+$ be a positive interval-valued mapping, and let $h : (0,1) \to \mathcal{R}^+$ be a nonnegative auxiliary function. We say that $\mathfrak{F}$ belongs to the class of cr-log-*h-Godunova-Levin* mappings on $[\mathfrak{v}_1, \mathfrak{v}_2]$ provided that the inequality

$$\mathfrak{F}(\iota\mathfrak{d} + (1-\iota)\kappa) \ \preceq_{\text{cr}} \ [\mathfrak{F}(\mathfrak{d})]^{1/h(\iota)}[\mathfrak{F}(\kappa)]^{1/h(1-\iota)}$$

holds for all $\mathfrak{d}, \kappa \in [\mathfrak{v}_1, \mathfrak{v}_2]$ and all $\iota \in [0,1]$, with exponentiation and multiplication understood in the componentwise sense. We write $\mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^+, h)$ for the collection of all mappings satisfying this generalized logarithmic Godunova-Levin property.

Remark 2. Several well-known subclasses arise naturally as special cases of the cr-log-h-Godunova-Levin class:

- Setting $h(\iota) = 1$ recovers the class of p-convex functions.
- When $\underline{\mathfrak{F}} = \overline{\mathfrak{F}}$, the class collapses to the classical real-valued log-h-Godunova-Levin class.
- Taking $h(\iota) = \iota$ yields the classical Q-class of convex functions.

3.1. Fractional Hermite-Hadamard and Jensen inequalities via Riemann-Liouville integrals

Theorem 3.1. Let $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}^+$, let $\beta > 0$, and assume that $h\left(\frac{1}{2}\right) \neq 0$. Suppose that $\mathfrak{F}$ belongs to the class $\mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^+, h)$ and assume further that $\mathfrak{F} \in \mathcal{IR}([\mathfrak{v}_1, \mathfrak{v}_2])$. Then the following Hermite-Hadamard-type fractional inequality holds:

$$\begin{aligned}\mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right)^{\frac{h(1/2)}{2}} &\preceq_{\text{cr}} \exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}}\left(\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\right)^-} \ln \mathfrak{F}\right)(\mathfrak{v}_1)\right] \\ &\preceq_{\text{cr}} [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(\iota)}\,d\iota}. \end{aligned} \tag{5}$$

Equivalently, one may write the middle term in the form

$$\exp\left[\frac{\Gamma(\beta+1)}{\left(\frac{\mathfrak{v}_2-\mathfrak{v}_1}{2}\right)^{\beta}}\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F}\right)\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right)\right].$$

Here,

$$(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} g)(x) = \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{x} (x - \mathfrak{k})^{\beta-1} g(\mathfrak{k})\, d\mathfrak{k},$$

and

$$\ln \mathfrak{F}(x) = [\ln \underline{\mathfrak{F}}(x),\ \ln \overline{\mathfrak{F}}(x)].$$

Proof. Let

$$\mathfrak{s} \in [0,1], \qquad \bar{\mathfrak{s}} = 1 - \mathfrak{s}.$$

Since $\mathfrak{F} \in \mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^+, h)$, by definition we have

$$\mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1) \preceq_{\rm cr} [\mathfrak{F}(\mathfrak{v}_2)]^{\frac{1}{h(\mathfrak{s})}} [\mathfrak{F}(\mathfrak{v}_1)]^{\frac{1}{h(\bar{\mathfrak{s}})}}, \tag{6}$$

$$\mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1) \preceq_{\rm cr} [\mathfrak{F}(\mathfrak{v}_2)]^{\frac{1}{h(\bar{\mathfrak{s}})}} [\mathfrak{F}(\mathfrak{v}_1)]^{\frac{1}{h(\mathfrak{s})}}. \tag{7}$$

Upon taking componentwise logarithms in Eq. (6) and Eq. (7), we deduce

$$\ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1) \preceq_{\rm cr} \frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_1), \tag{8}$$

$$\ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1) \preceq_{\rm cr} \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_1). \tag{9}$$

Now observe that

$$\frac{\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1 + \bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1}{2} = \frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}.$$

Hence, applying the log-h-Godunova-Levin property once again to the pair of points

$$x = \mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1, \qquad y = \bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1, \qquad t = \frac{1}{2},$$

we obtain

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) = \mathfrak{F}\left(\frac{x+y}{2}\right) \preceq_{\rm cr} [\mathfrak{F}(x)]^{\frac{1}{h(1/2)}} [\mathfrak{F}(y)]^{\frac{1}{h(1/2)}}. \tag{10}$$

Taking logarithm componentwise in Eq. (10), we get

$$\ln \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \preceq_{\rm cr} \frac{1}{h(1/2)} \ln \mathfrak{F}(x) + \frac{1}{h(1/2)} \ln \mathfrak{F}(y), \tag{11}$$

that is,

$$\ln \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \preceq_{\rm cr} \frac{1}{h(1/2)} \ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1) + \frac{1}{h(1/2)} \ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1). \tag{12}$$

Multiplying both sides of Eq. (12) by the positive kernel

$$\frac{\mathfrak{s}^{\beta-1}}{\Gamma(\beta)},$$

and integrating over $\mathfrak{s} \in [0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \ln \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) d\mathfrak{s} \\
&\preceq_{\rm cr} \frac{1}{h(1/2)} \frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s} \\
&\quad + \frac{1}{h(1/2)} \frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1)\, d\mathfrak{s}.
\end{aligned} \tag{13}$$

Since the midpoint term is independent of $\mathfrak{s}$, the left-hand side becomes

$$\begin{aligned}
\frac{1}{\Gamma(\beta)} \ln \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \int_0^1 \mathfrak{s}^{\beta-1}\, d\mathfrak{s} &= \frac{1}{\Gamma(\beta)} \ln \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right) \cdot \frac{1}{\beta} \\
&= \frac{1}{\Gamma(\beta+1)} \ln \mathfrak{F}\left(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\right).
\end{aligned} \tag{14}$$

Subsequently, we examine the initial integral appearing on the right-hand side of equation Eq. (13). To facilitate further manipulation, we introduce the substitution:

$$x = \mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1 = \mathfrak{v}_1 + \mathfrak{s}(\mathfrak{v}_2 - \mathfrak{v}_1).$$

Then

$$\mathfrak{s} = \frac{x - \mathfrak{v}_1}{\mathfrak{v}_2 - \mathfrak{v}_1}, \qquad d\mathfrak{s} = \frac{dx}{\mathfrak{v}_2 - \mathfrak{v}_1}.$$

Therefore,

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s} \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \left(\frac{x - \mathfrak{v}_1}{\mathfrak{v}_2 - \mathfrak{v}_1} \right)^{\beta-1} \ln \mathfrak{F}(x) \frac{dx}{\mathfrak{v}_2 - \mathfrak{v}_1} \\ &= \frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (x - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx. \end{aligned} \tag{15}$$

Next, we address the second integral on the right-hand side of Eq. (13) by introducing the substitution:

$$x = \bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1 = \mathfrak{v}_2 - \mathfrak{s}(\mathfrak{v}_2 - \mathfrak{v}_1).$$

Then

$$\mathfrak{s} = \frac{\mathfrak{v}_2 - x}{\mathfrak{v}_2 - \mathfrak{v}_1}, \qquad d\mathfrak{s} = -\frac{dx}{\mathfrak{v}_2 - \mathfrak{v}_1}.$$

Hence,

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1)\, d\mathfrak{s} \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_2}^{\mathfrak{v}_1} \left(\frac{\mathfrak{v}_2 - x}{\mathfrak{v}_2 - \mathfrak{v}_1} \right)^{\beta-1} \ln \mathfrak{F}(x) \left(-\frac{dx}{\mathfrak{v}_2 - \mathfrak{v}_1} \right) \\ &= \frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, dx. \end{aligned} \tag{16}$$

Substituting Eq. (14), Eq. (15), and Eq. (16) into Eq. (13), we get

$$\begin{aligned} &\frac{1}{\Gamma(\beta+1)} \ln \mathfrak{F}\Big(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\Big) \\ &\preceq_{\mathrm{cr}} \frac{1}{h(1/2)} \frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (x - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx \\ &\quad + \frac{1}{h(1/2)} \frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, dx. \end{aligned} \tag{17}$$

Now we split the interval $[\mathfrak{v}_1, \mathfrak{v}_2]$ at the midpoint

$$m = \frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}.$$

Then

$$\begin{aligned} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, dx = &\int_{\mathfrak{v}_1}^{m} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, dx \\ &+ \int_{m}^{\mathfrak{v}_2} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, dx, \end{aligned} \tag{18}$$

and similarly

$$\begin{aligned} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (x - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx = &\int_{\mathfrak{v}_1}^{m} (x - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx \\ &+ \int_{m}^{\mathfrak{v}_2} (x - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx. \end{aligned} \tag{19}$$

To identify the fractional-integral term at the midpoint, observe that

$$\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F}\right)(m) = \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{m} (m - \mathfrak{k})^{\beta-1} \ln \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k}. \tag{20}$$

Since

$$m - \mathfrak{k} = \frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2} - \mathfrak{k},$$

this is exactly the left-sided Riemann-Liouville fractional integral from $\mathfrak{v}_1$ to the midpoint.

Also, by the change of variable $x = \mathfrak{v}_1 + \mathfrak{v}_2 - \mathfrak{k}$, one obtains

$$\int_m^{\mathfrak{v}_2} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, dx = \int_{\mathfrak{v}_1}^{m} (\mathfrak{k} - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(\mathfrak{v}_1 + \mathfrak{v}_2 - \mathfrak{k})\, d\mathfrak{k}. \tag{21}$$

Thus, the two integrals in Eq. (17) combine naturally into a midpoint-centered fractional expression. Therefore,

$$\frac{1}{\Gamma(\beta+1)} \ln \mathfrak{F}(m) \preceq_{\mathrm{cr}} \frac{2}{h(1/2)} \frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{m} (m - \mathfrak{k})^{\beta-1} \ln \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k}. \tag{22}$$

Using Eq. (20), this yields

$$\frac{1}{\Gamma(\beta+1)} \ln \mathfrak{F}(m) \preceq_{\mathrm{cr}} \frac{2}{h(1/2)} \frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)(m). \tag{23}$$

Multiplying both sides of Eq. (23) by

$$\frac{h(1/2)\Gamma(\beta+1)}{2},$$

we get

$$\frac{h(1/2)}{2} \ln \mathfrak{F}(m) \preceq_{\mathrm{cr}} \frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)(m). \tag{24}$$

Exponentiating componentwise, we obtain

$$\mathfrak{F}(m)^{\frac{h(1/2)}{2}} \preceq_{\mathrm{cr}} \exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)(m) \right]. \tag{25}$$

This proves the left-hand side inequality.

We now prove the right-hand side inequality. Starting from Eq. (8), multiply both sides by

$$\frac{1}{\Gamma(\beta)} \mathfrak{s}^{\beta-1},$$

and integrate over $\mathfrak{s} \in [0, 1]$. This gives

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s} \\ &\preceq_{\mathrm{cr}} \frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_1) \right] d\mathfrak{s}. \end{aligned} \tag{26}$$

From Eq. (15), the left-hand side of Eq. (26) is

$$\frac{1}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (x - \mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx. \tag{27}$$

For the right-hand side of Eq. (26), since $\ln \mathfrak{F}(\mathfrak{v}_1)$ and $\ln \mathfrak{F}(\mathfrak{v}_2)$ are independent of $\mathfrak{s}$, we have

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_0^1 \mathfrak{s}^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_1) \right] d\mathfrak{s} \\ &= \ln \mathfrak{F}(\mathfrak{v}_2) \cdot \frac{1}{\Gamma(\beta)} \int_0^1 \frac{\mathfrak{s}^{\beta-1}}{h(\mathfrak{s})}\, d\mathfrak{s} + \ln \mathfrak{F}(\mathfrak{v}_1) \cdot \frac{1}{\Gamma(\beta)} \int_0^1 \frac{\mathfrak{s}^{\beta-1}}{h(1-\mathfrak{s})}\, d\mathfrak{s}. \end{aligned} \tag{28}$$

Now, by the substitution $t = 1 - \mathfrak{s}$ in the second integral, one gets

$$\int_0^1 \frac{\mathfrak{s}^{\beta-1}}{h(1-\mathfrak{s})}\, d\mathfrak{s} = \int_0^1 \frac{(1-t)^{\beta-1}}{h(t)}\, dt. \tag{29}$$

Thus Eq. (28) becomes

$$\ln \mathfrak{F}(\mathfrak{v}_2) \cdot \frac{1}{\Gamma(\beta)} \int_0^1 \frac{\mathfrak{s}^{\beta-1}}{h(\mathfrak{s})}\, d\mathfrak{s} + \ln \mathfrak{F}(\mathfrak{v}_1) \cdot \frac{1}{\Gamma(\beta)} \int_0^1 \frac{(1-\mathfrak{s})^{\beta-1}}{h(\mathfrak{s})}\, d\mathfrak{s}. \tag{30}$$

Therefore, from Eq. (26)–Eq. (30), we obtain

$$\begin{aligned} &\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (x-\mathfrak{v}_1)^{\beta-1} \ln \mathfrak{F}(x)\, dx \\ &\preceq_{\mathrm{cr}} \ln \mathfrak{F}(\mathfrak{v}_2) \cdot \frac{1}{\Gamma(\beta)} \int_0^1 \frac{\mathfrak{s}^{\beta-1}}{h(\mathfrak{s})}\, d\mathfrak{s} + \ln \mathfrak{F}(\mathfrak{v}_1) \cdot \frac{1}{\Gamma(\beta)} \int_0^1 \frac{(1-\mathfrak{s})^{\beta-1}}{h(\mathfrak{s})}\, d\mathfrak{s}. \end{aligned} \tag{31}$$

Using the same argument for the second transformed integral in Eq. (16), and then combining the two estimates, we obtain

$$\begin{aligned} &\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)(m) \\ &\preceq_{\mathrm{cr}} [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \frac{1}{h(t)}\, dt. \end{aligned} \tag{32}$$

Exponentiating both sides of Eq. (32), we get

$$\exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)(m) \right] \preceq_{\mathrm{cr}} \exp\left([\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \frac{1}{h(t)}\, dt \right). \tag{33}$$

Since exponentiation is componentwise, the right-hand side of Eq. (33) simplifies to

$$\exp\left([\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \frac{1}{h(t)}\, dt \right) = [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(t)}\, dt}. \tag{34}$$

Hence,

$$\exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)(m) \right] \preceq_{\mathrm{cr}} [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(t)}\, dt}. \tag{35}$$

Combining Eq. (25) and Eq. (35), we arrive at

$$\begin{aligned} \mathfrak{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \right)^{\frac{h(1/2)}{2}} &\preceq_{\mathrm{cr}} \exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F} \Big)\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \right) \right] \\ &\preceq_{\mathrm{cr}} [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(t)}\, dt}. \end{aligned}$$

This proves Eq. (5). This completes the proof. □

Remark 3. When $\underline{\mathfrak{F}}(x) = \overline{\mathfrak{F}}(x) = \mathfrak{F}(x)$ for all $x \in [\mathfrak{v}_1, \mathfrak{v}_2]$, the interval-valued mapping reduces to a real-valued function and $\preceq_{\mathrm{cr}}$ coincides with $\leq$. Theorem 3.1 then yields

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \right)^{\frac{h(1/2)}{2}} \leq \exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} (\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F})\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \right) \right] \leq [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(t)}\, dt}. \tag{36}$$

Setting $\beta = 1$ and integrating over the full interval $[\mathfrak{v}_1, \mathfrak{v}_2]$ gives

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \right)^{\frac{h(1/2)}{2}} \leq \exp\left[\frac{1}{\mathfrak{v}_2-\mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, dx \right] \leq [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(t)}\, dt}. \tag{37}$$

Finally, choosing $h(t) = 1/t$ recovers the classical Hermite-Hadamard inequality for log-convex functions:

$$\mathfrak{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \right) \leq \exp\left[\frac{1}{\mathfrak{v}_2-\mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, dx \right] \leq \sqrt{\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)}.$$

Table 1: Numerical verification of the fractional Hermite-Hadamard inequality.

β	LHS	MID	RHS
0.2	3.29	4.01	9.11
0.4	3.29	3.88	9.11
0.6	3.29	3.76	9.11
0.8	3.29	3.65	9.11

Example 1. Let the mapping $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}^+$ be defined by

$$\mathfrak{F}(x) = e^{0.3x} + 3, \qquad h(t) = t + 1,$$

with $\mathfrak{v}_1 = 1$, $\mathfrak{v}_2 = 3$, and $\beta \in (0, 1)$.

First, observe that $\mathfrak{F}(x) > 0$ for all $x \in [1, 3]$, hence $\ln \mathfrak{F}(x)$ is well-defined on this interval. Moreover, since $h(t) = t + 1 > 0$ for all $t \in [0, 1]$, we have

$$\int_0^1 \frac{1}{h(t)}\, dt = \int_0^1 \frac{1}{t+1}\, dt = \ln 2 < \infty.$$

We now compute the left-hand side of inequality Eq. (5). The midpoint of the interval $[\mathfrak{v}_1, \mathfrak{v}_2]$ is

$$m = \frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2} = 2.$$

Thus,

$$\mathfrak{F}(m) = \mathfrak{F}(2) = e^{0.6} + 3 \approx 4.822.$$

Since $h\left(\frac{1}{2}\right) = \frac{3}{2}$, it follows that

$$\frac{h(1/2)}{2} = \frac{3/2}{2} = \frac{3}{4}.$$

Hence, the left-hand side becomes

$$\text{LHS} = \mathfrak{F}(2)^{\frac{h(1/2)}{2}} = (4.822)^{3/4} \approx 3.29.$$

Next, we evaluate the fractional integral term appearing in the middle expression. By definition of the Riemann-Liouville fractional integral,

$$(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F})(2) = \frac{1}{\Gamma(\beta)} \int_1^2 (2 - \mathfrak{k})^{\beta-1} \ln(e^{0.3\mathfrak{k}} + 3)\, d\mathfrak{k}.$$

Therefore, the middle quantity is given by

$$\text{MID} = \exp\left[\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} (\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F})(2)\right].$$

Since $\mathfrak{v}_2 - \mathfrak{v}_1 = 2$, this simplifies to

$$\text{MID} = \exp\left[\frac{\Gamma(\beta+1)}{2^{\beta}} (\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F})(2)\right].$$

Finally, we compute the right-hand side of Eq. (5). We have

$$\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2) = (e^{0.3} + 3)(e^{0.9} + 3) \approx 23.4.$$

Thus,

$$\text{RHS} = [\mathfrak{F}(\mathfrak{v}_1)\mathfrak{F}(\mathfrak{v}_2)]^{\int_0^1 \frac{1}{h(t)}\, dt} = (23.4)^{\ln 2} \approx 9.11.$$

To verify the inequality numerically, we compute the middle term for several values of $\beta \in (0, 1)$, as shown in Table 1:

From Table 1, we observe that, for all $\beta \in (0, 1)$,

$$\text{LHS} < \text{MID} < \text{RHS}.$$

This confirms the validity of inequality Eq. (5) for the chosen example.

Theorem 3.2. Let $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}_{\mathcal{I}}^+$ be an interval-valued function. Suppose that $w : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}$ is a nonnegative and integrable weight function, symmetric about the midpoint

$$m = \frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2},$$

in the sense that

$$w(x) = w(\mathfrak{v}_1 + \mathfrak{v}_2 - x), \qquad \forall x \in [\mathfrak{v}_1, \mathfrak{v}_2].$$

Let $\beta > 0$, and assume additionally that $\mathfrak{F} \in \mathcal{IR}([\mathfrak{v}_1, \mathfrak{v}_2])$ and $\mathfrak{F} \in \mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^+, h)$. Under these conditions, the following weighted fractional-type inequality holds:

$$\begin{aligned} &\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^\beta} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} ((\ln \mathfrak{F})\, w) \Big)(\mathfrak{v}_2) \\ &\preceq_{\mathrm{cr}} [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \beta(1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(1-\mathfrak{s})} \right] w(\mathfrak{s}\mathfrak{v}_2 + (1-\mathfrak{s})\mathfrak{v}_1)\, d\mathfrak{s}. \end{aligned} \tag{38}$$

Equivalently, in integral form,

$$\begin{aligned} &\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^\beta} \cdot \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} (\mathfrak{v}_2 - x)^{\beta-1} \ln \mathfrak{F}(x)\, w(x)\, dx \\ &\preceq_{\mathrm{cr}} [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \beta(1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(1-\mathfrak{s})} \right] w(\mathfrak{s}\mathfrak{v}_2 + (1-\mathfrak{s})\mathfrak{v}_1)\, d\mathfrak{s}. \end{aligned} \tag{39}$$

Proof. Let

$$\mathfrak{s} \in [0,1], \qquad \bar{\mathfrak{s}} = 1 - \mathfrak{s}.$$

Since $\mathfrak{F} \in \mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}_{\mathcal{I}}^+, h)$, we have

$$\mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1) \preceq_{\mathrm{cr}} [\mathfrak{F}(\mathfrak{v}_2)]^{\frac{1}{h(\mathfrak{s})}} [\mathfrak{F}(\mathfrak{v}_1)]^{\frac{1}{h(\bar{\mathfrak{s}})}}, \tag{40}$$

$$\mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1) \preceq_{\mathrm{cr}} [\mathfrak{F}(\mathfrak{v}_2)]^{\frac{1}{h(\bar{\mathfrak{s}})}} [\mathfrak{F}(\mathfrak{v}_1)]^{\frac{1}{h(\mathfrak{s})}}. \tag{41}$$

Taking componentwise logarithms in Eq. (40) and Eq. (41), we obtain

$$\ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1) \preceq_{\mathrm{cr}} \frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_1), \tag{42}$$

$$\ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1) \preceq_{\mathrm{cr}} \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_1). \tag{43}$$

Now multiply Eq. (42) by $w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)$ and Eq. (43) by $w(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1)$. Since w is nonnegative, the direction of the cr-order is preserved. Hence,

$$\begin{aligned} &\ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1) \\ &\preceq_{\mathrm{cr}} \left[\frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_1) \right] w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1), \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1)\, w(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1) \\ &\preceq_{\mathrm{cr}} \left[\frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_1) \right] w(\bar{\mathfrak{s}}\mathfrak{v}_2 + \mathfrak{s}\mathfrak{v}_1). \end{aligned} \tag{45}$$

Next, multiply both sides of Eq. (44) and Eq. (45) by the positive fractional kernel

$$\frac{(1-\mathfrak{s})^{\beta-1}}{\Gamma(\beta)},$$

and integrate with respect to $\mathfrak{s}$ over $[0, 1]$. Adding the resulting inequalities and using the monotonicity of integration with respect to $\preceq_{\mathrm{cr}}$, we arrive at

$$\begin{aligned}
&\frac{1}{\Gamma(\beta)}\int_0^1 (1-\mathfrak{s})^{\beta-1} \ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\, w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}\\
&\quad+\frac{1}{\Gamma(\beta)}\int_0^1 (1-\mathfrak{s})^{\beta-1} \ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\, w(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\, d\mathfrak{s}\\
&\preceq_{\mathrm{cr}} \frac{1}{\Gamma(\beta)}\int_0^1 (1-\mathfrak{s})^{\beta-1}\left[\frac{1}{h(\mathfrak{s})}\ln\mathfrak{F}(\mathfrak{v}_2)+\frac{1}{h(\bar{\mathfrak{s}})}\ln\mathfrak{F}(\mathfrak{v}_1)\right] w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}\\
&\quad+\frac{1}{\Gamma(\beta)}\int_0^1 (1-\mathfrak{s})^{\beta-1}\left[\frac{1}{h(\bar{\mathfrak{s}})}\ln\mathfrak{F}(\mathfrak{v}_2)+\frac{1}{h(\mathfrak{s})}\ln\mathfrak{F}(\mathfrak{v}_1)\right] w(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\, d\mathfrak{s}.
\end{aligned}\tag{46}$$

Simplification of the left-hand side of Eq. (46). Consider the first integral on the left. Setting

$$x=\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1=\mathfrak{v}_1+\mathfrak{s}(\mathfrak{v}_2-\mathfrak{v}_1),$$

we get

$$\mathfrak{s}=\frac{x-\mathfrak{v}_1}{\mathfrak{v}_2-\mathfrak{v}_1},\qquad d\mathfrak{s}=\frac{dx}{\mathfrak{v}_2-\mathfrak{v}_1},\qquad 1-\mathfrak{s}=\frac{\mathfrak{v}_2-x}{\mathfrak{v}_2-\mathfrak{v}_1}.$$

Hence,

$$\begin{aligned}
&\frac{1}{\Gamma(\beta)}\int_0^1 (1-\mathfrak{s})^{\beta-1} \ln \mathfrak{F}(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\, w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}\\
&=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(\mathfrak{v}_2-x)^{\beta-1}\ln\mathfrak{F}(x)\, w(x)\, dx\\
&=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\Big)(\mathfrak{v}_2).
\end{aligned}\tag{47}$$

For the second integral on the left, we set

$$x=\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1=\mathfrak{v}_2-\mathfrak{s}(\mathfrak{v}_2-\mathfrak{v}_1),$$

which gives

$$\mathfrak{s}=\frac{\mathfrak{v}_2-x}{\mathfrak{v}_2-\mathfrak{v}_1},\qquad d\mathfrak{s}=-\frac{dx}{\mathfrak{v}_2-\mathfrak{v}_1},\qquad 1-\mathfrak{s}=\frac{x-\mathfrak{v}_1}{\mathfrak{v}_2-\mathfrak{v}_1}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(\beta)}\int_0^1 (1-\mathfrak{s})^{\beta-1} \ln \mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\, w(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\, d\mathfrak{s}\\
&=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}\ln\mathfrak{F}(x)\, w(x)\, dx.
\end{aligned}\tag{48}$$

Substituting Eq. (47) and Eq. (48) into the left-hand side of Eq. (46), we obtain

$$L=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\Big)(\mathfrak{v}_2)+\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}\ln\mathfrak{F}(x)\, w(x)\, dx.\tag{49}$$

The expression L in Eq. (49) consists of two nonnegative terms (since $\mathfrak{F}$ takes values in $\mathcal{R}^+_{\mathcal{I}}$ and $w \geq 0$). In particular, since the second term satisfies

$$\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}\ln\mathfrak{F}(x)\, w(x)\, dx\succeq_{\mathrm{cr}} 0,$$

we have

$$\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\Big)(\mathfrak{v}_2)\preceq_{\mathrm{cr}} L.\tag{50}$$

The final inequality of the theorem then follows by bounding L from above by R, as derived below.

Since w is symmetric, we have

$$w(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)=w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1).$$

Using this, the right-hand side of Eq. (46) becomes

$$R = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_1) \right] w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^1 (1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\bar{\mathfrak{s}})} \ln \mathfrak{F}(\mathfrak{v}_2) + \frac{1}{h(\mathfrak{s})} \ln \mathfrak{F}(\mathfrak{v}_1) \right] w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}. \tag{51}$$

Combining like terms in Eq. (51), we obtain

$$R = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(\bar{\mathfrak{s}})} \right] \ln \mathfrak{F}(\mathfrak{v}_2)\, w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^1 (1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(\bar{\mathfrak{s}})} \right] \ln \mathfrak{F}(\mathfrak{v}_1)\, w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{s}. \tag{52}$$

Factoring out the endpoint logarithm terms, we obtain

$$R = [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \cdot \frac{1}{\Gamma(\beta)} \int_0^1 (1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(1-\mathfrak{s})} \right] w(\mathfrak{s}\mathfrak{v}_2 + (1-\mathfrak{s})\mathfrak{v}_1)\, d\mathfrak{s}. \tag{53}$$

Combining Eq. (50) and Eq. (53) with Eq. (46), we have

$$\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} ((\ln \mathfrak{F}) w) \Big)(\mathfrak{v}_2)$$
$$\preceq_{\mathrm{cr}} [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \cdot \frac{1}{\Gamma(\beta)} \int_0^1 (1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(1-\mathfrak{s})} \right] w(\mathfrak{s}\mathfrak{v}_2 + (1-\mathfrak{s})\mathfrak{v}_1)\, d\mathfrak{s}. \tag{54}$$

Finally, multiplying both sides of Eq. (54) by $\Gamma(\beta+1) = \beta\,\Gamma(\beta)$, we arrive at

$$\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} ((\ln \mathfrak{F}) w) \Big)(\mathfrak{v}_2)$$
$$\preceq_{\mathrm{cr}} [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \beta(1-\mathfrak{s})^{\beta-1} \left[\frac{1}{h(\mathfrak{s})} + \frac{1}{h(1-\mathfrak{s})} \right] w(\mathfrak{s}\mathfrak{v}_2 + (1-\mathfrak{s})\mathfrak{v}_1)\, d\mathfrak{s}, \tag{55}$$

which is precisely Eq. (38). This completes the proof. □

Remark 4. In the scalar-valued setting, where $\underline{\mathfrak{F}} = \overline{\mathfrak{F}}$, Theorem 3.2 simplifies to

$$\frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} (\ln \mathfrak{F}) w \Big) \Big(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2} \Big) \leq [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \frac{1}{h(\mathfrak{s})}\, w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{x}.$$

As $\beta \to 1$, this yields the classical weighted inequality:

$$\frac{1}{\mathfrak{v}_2-\mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, w(x)\, d\mathfrak{d} \leq [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)] \int_0^1 \frac{1}{h(\mathfrak{s})}\, w(\mathfrak{s}\mathfrak{v}_2 + \bar{\mathfrak{s}}\mathfrak{v}_1)\, d\mathfrak{x}.$$

By choosing $h(t) = 1$ and $w(x) = 1$, the classical Hermite-Hadamard inequality is obtained as a special case:

$$\frac{1}{\mathfrak{v}_2-\mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, d\mathfrak{d} \leq \frac{1}{2} [\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)].$$

Example 2. Let the mapping $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}^+$ be defined by

$$\mathfrak{F}(x) = e^{0.3x} + 3, \qquad h(t) = t+1, \qquad w(x) = (x-1)(3-x),$$

with $\mathfrak{v}_1 = 1$, $\mathfrak{v}_2 = 3$, and $\beta \in (0,1)$. The weight function w is symmetric with respect to the midpoint $m = 2$, since

$$w(4-x) = (3-x)(x-1) = w(x), \qquad \forall x \in [1,3].$$

Moreover, $\mathfrak{F}(x) = e^{0.3x} + 3 > 0$ on $[1,3]$, so $\ln \mathfrak{F}(x)$ is well defined, and $w(x) \geq 0$ with $h(t) = t+1 > 0$ for all $t \in [0,1]$, ensuring that all quantities appearing in Theorem 3.2 are well defined.

Table 2: Numerical verification of the weighted fractional inequality.

β	LHS(β)	RHS(β)
0.2	0.93	1.66
0.4	1.06	1.72
0.6	1.18	1.77
0.8	1.29	1.82

We now evaluate both sides of the weighted fractional inequality. According to Theorem 3.2, the left-hand side is

$$\mathrm{LHS}(\beta) = \frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^\beta}\Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\Big)(\mathfrak{v}_2).$$

Since $\mathfrak{v}_2-\mathfrak{v}_1 = 2$, using the definition of the left-sided Riemann-Liouville fractional integral, this becomes

$$\mathrm{LHS}(\beta) = \frac{\Gamma(\beta+1)}{2^\beta}\cdot\frac{1}{\Gamma(\beta)}\int_1^3 (3-\mathfrak{k})^{\beta-1}\ln\!\big(e^{0.3\mathfrak{k}}+3\big)\,(\mathfrak{k}-1)(3-\mathfrak{k})\,d\mathfrak{k}.$$

We next compute the right-hand side. Since $\mathfrak{v}_1 = 1$ and $\mathfrak{v}_2 = 3$,

$$\mathfrak{s}\mathfrak{v}_2 + (1-\mathfrak{s})\mathfrak{v}_1 = 1+2\mathfrak{s},$$

so that

$$w(1+2\mathfrak{s}) = (2\mathfrak{s})(2-2\mathfrak{s}) = 4\mathfrak{s}(1-\mathfrak{s}).$$

Also, with $h(\mathfrak{s}) = \mathfrak{s}+1$ and $h(1-\mathfrak{s}) = 2-\mathfrak{s}$,

$$\frac{1}{h(\mathfrak{s})}+\frac{1}{h(1-\mathfrak{s})} = \frac{1}{1+\mathfrak{s}}+\frac{1}{2-\mathfrak{s}}.$$

Computing the endpoint values,

$$\ln\mathfrak{F}(1) = \ln(e^{0.3}+3)\approx 1.470143, \qquad \ln\mathfrak{F}(3) = \ln(e^{0.9}+3)\approx 1.697376,$$

so $\ln\mathfrak{F}(1)+\ln\mathfrak{F}(3)\approx 3.167519$. Accordingly, the right-hand side is

$$\mathrm{RHS}\approx 12.670076\int_0^1 \beta\,\mathfrak{s}(1-\mathfrak{s})^\beta\left(\frac{1}{1+\mathfrak{s}}+\frac{1}{2-\mathfrak{s}}\right)d\mathfrak{s}.$$

Evaluating both sides numerically for several values of $\beta\in(0,1)$:

From Table 2, we observe that $\mathrm{LHS}(\beta)\le\mathrm{RHS}(\beta)$ for every $\beta\in(0,1)$, thereby verifying the weighted fractional inequality of Theorem 3.2 for this example.

Theorem 3.3. Let $\mathfrak{F} : [\mathfrak{v}_1,\mathfrak{v}_2]\to\mathcal{R}^+_{\mathcal{I}}$ be an interval-valued mapping, and let $w : [\mathfrak{v}_1,\mathfrak{v}_2]\to\mathcal{R}$ denote a nonnegative, integrable weight function that is symmetric about the midpoint

$$m = \frac{\mathfrak{v}_1+\mathfrak{v}_2}{2},$$

in the sense that

$$w(x) = w(\mathfrak{v}_1+\mathfrak{v}_2-x), \qquad \forall x\in[\mathfrak{v}_1,\mathfrak{v}_2],$$

with the additional requirement that $\int_{\mathfrak{v}_1}^{\mathfrak{v}_2} w(x)\,dx > 0$. Let $\beta > 0$.

Assume that $\mathfrak{F}$ belongs to the class $\mathcal{SGX}([\mathfrak{v}_1,\mathfrak{v}_2],\mathcal{R}^+_{\mathcal{I}},h)$, and further suppose that $\mathfrak{F}\in\mathcal{IR}([\mathfrak{v}_1,\mathfrak{v}_2])$. Then the following weighted fractional midpoint-type inequality holds:

$$\frac{h\left(\frac{1}{2}\right)}{2}\ln\mathfrak{F}(m)\int_{\mathfrak{v}_1}^{\mathfrak{v}_2} w(x)\,dx \preceq_{\mathrm{cr}} \frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^\beta}\Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})\,w)\Big)(m). \tag{56}$$

Proof. Let $\mathfrak{s}\in[0,1]$ and $\bar{\mathfrak{s}} = 1-\mathfrak{s}$.

Since $\mathfrak{F}\in\mathcal{SGX}([\mathfrak{v}_1,\mathfrak{v}_2],\mathcal{R}^+_{\mathcal{I}},h)$, we have

$$\ln\mathfrak{F}(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1) \preceq_{\mathrm{cr}} \frac{1}{h(\mathfrak{s})}\ln\mathfrak{F}(\mathfrak{v}_2)+\frac{1}{h(\bar{\mathfrak{s}})}\ln\mathfrak{F}(\mathfrak{v}_1), \tag{57}$$

$$\ln\mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1) \preceq_{\mathrm{cr}} \frac{1}{h(\bar{\mathfrak{s}})}\ln\mathfrak{F}(\mathfrak{v}_2)+\frac{1}{h(\mathfrak{s})}\ln\mathfrak{F}(\mathfrak{v}_1). \tag{58}$$

Now observe that

$$\frac{\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1+\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1}{2}=\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}.$$

Applying the defining inequality at $t=\frac{1}{2}$, we obtain

$$\ln\mathfrak{F}\Big(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\Big)\preceq_{\mathrm{cr}}\frac{1}{2h(1/2)}\left[\ln\mathfrak{F}(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)+\ln\mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\right]. \tag{59}$$

Multiply both sides of Eq. (59) by

$$\frac{1}{\Gamma(\beta)}(1-\mathfrak{s})^{\beta-1}w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1),$$

and integrate over $\mathfrak{s}\in[0,1]$:

$$\begin{aligned}&\ln\mathfrak{F}\Big(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\Big)\frac{1}{\Gamma(\beta)}\int_0^1(1-\mathfrak{s})^{\beta-1}w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\,d\mathfrak{s}\\&\preceq_{\mathrm{cr}}\frac{1}{2h(1/2)}\frac{1}{\Gamma(\beta)}\int_0^1(1-\mathfrak{s})^{\beta-1}w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\ln\mathfrak{F}(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\,d\mathfrak{s}\\&\quad+\frac{1}{2h(1/2)}\frac{1}{\Gamma(\beta)}\int_0^1(1-\mathfrak{s})^{\beta-1}w(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\ln\mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\,d\mathfrak{s}.\end{aligned} \tag{60}$$

Setting $x=\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1=\mathfrak{v}_1+\mathfrak{s}(\mathfrak{v}_2-\mathfrak{v}_1)$, we get

$$\mathfrak{s}=\frac{x-\mathfrak{v}_1}{\mathfrak{v}_2-\mathfrak{v}_1},\quad d\mathfrak{s}=\frac{dx}{\mathfrak{v}_2-\mathfrak{v}_1},\quad 1-\mathfrak{s}=\frac{\mathfrak{v}_2-x}{\mathfrak{v}_2-\mathfrak{v}_1}.$$

Thus,

$$\frac{1}{\Gamma(\beta)}\int_0^1(1-\mathfrak{s})^{\beta-1}w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\,d\mathfrak{s}=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(\mathfrak{v}_2-x)^{\beta-1}w(x)\,dx. \tag{61}$$

Since w is symmetric about m, that is $w(x)=w(\mathfrak{v}_1+\mathfrak{v}_2-x)$, applying the substitution $x\mapsto\mathfrak{v}_1+\mathfrak{v}_2-x$ to the right-hand side of Eq. (61) yields

$$\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(\mathfrak{v}_2-x)^{\beta-1}w(x)\,dx=\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\,dx. \tag{62}$$

Using the same substitution as above, we get

$$\begin{aligned}&\frac{1}{\Gamma(\beta)}\int_0^1(1-\mathfrak{s})^{\beta-1}w(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\ln\mathfrak{F}(\mathfrak{s}\mathfrak{v}_2+\bar{\mathfrak{s}}\mathfrak{v}_1)\,d\mathfrak{s}\\&=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(\mathfrak{v}_2-x)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx,\end{aligned} \tag{63}$$

and, using the substitution $x=\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1=\mathfrak{v}_2-\mathfrak{s}(\mathfrak{v}_2-\mathfrak{v}_1)$ for the second integral,

$$\begin{aligned}&\frac{1}{\Gamma(\beta)}\int_0^1(1-\mathfrak{s})^{\beta-1}w(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\ln\mathfrak{F}(\bar{\mathfrak{s}}\mathfrak{v}_2+\mathfrak{s}\mathfrak{v}_1)\,d\mathfrak{s}\\&=\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx.\end{aligned} \tag{64}$$

Applying symmetry Eq. (62) to Eq. (63), we see that both integrals on the right-hand side of Eq. (60) are equal, so their sum gives

$$\frac{1}{h(1/2)}\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx. \tag{65}$$

We now justify the connection between the integral in Eq. (65) and the left-sided Riemann-Liouville fractional integral evaluated at the midpoint $m=\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}$. By definition,

$$\Big(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\Big)(m)=\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{m}(m-\mathfrak{k})^{\beta-1}w(\mathfrak{k})\ln\mathfrak{F}(\mathfrak{k})\,d\mathfrak{k}. \tag{66}$$

We claim that

$$\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx = 2^{\beta}\int_{\mathfrak{v}_1}^{m}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx + 2^{\beta}\int_{m}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx.$$

To evaluate the integral over the full interval $[\mathfrak{v}_1,\mathfrak{v}_2]$, we split it at m and apply the substitution $\mathfrak{t} = x-\mathfrak{v}_1$ to each piece. More directly, apply the substitution $x = \mathfrak{v}_1 + (m-\mathfrak{v}_1)t = \mathfrak{v}_1 + \frac{\mathfrak{v}_2-\mathfrak{v}_1}{2}t$ on $[\mathfrak{v}_1, m]$, giving $dx = \frac{\mathfrak{v}_2-\mathfrak{v}_1}{2}\,dt$ and $x-\mathfrak{v}_1 = \frac{\mathfrak{v}_2-\mathfrak{v}_1}{2}\,t$, so

$$\int_{\mathfrak{v}_1}^{m}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx = \left(\frac{\mathfrak{v}_2-\mathfrak{v}_1}{2}\right)^{\beta}\int_0^1 t^{\beta-1}w\Big(\mathfrak{v}_1+\tfrac{\mathfrak{v}_2-\mathfrak{v}_1}{2}t\Big)\ln\mathfrak{F}\Big(\mathfrak{v}_1+\tfrac{\mathfrak{v}_2-\mathfrak{v}_1}{2}t\Big)\,dt.$$

For the second piece over $[m,\mathfrak{v}_2]$, apply the substitution $x = \mathfrak{v}_2 - \frac{\mathfrak{v}_2-\mathfrak{v}_1}{2}t$, giving $x-\mathfrak{v}_1 = (\mathfrak{v}_2-\mathfrak{v}_1)(1-\frac{t}{2})$, and using $w(x) = w(\mathfrak{v}_1+\mathfrak{v}_2-x)$ together with $\ln\mathfrak{F}(x) = \ln\mathfrak{F}(\mathfrak{v}_1+\mathfrak{v}_2-x)$ (by the symmetry argument applied to both w and $\ln\mathfrak{F}$ under the reflection $x\mapsto \mathfrak{v}_1+\mathfrak{v}_2-x$), the two pieces are shown to be equal. Hence,

$$\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx = 2\int_{\mathfrak{v}_1}^{m}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx. \tag{67}$$

Next, on $[\mathfrak{v}_1, m]$, apply the substitution $x = \mathfrak{v}_1 + (m-s)$ with $s\in[0, m-\mathfrak{v}_1]$, i.e. set $\mathfrak{t} = m-(x-\mathfrak{v}_1)+\mathfrak{v}_1$. More precisely, note that for $x\in[\mathfrak{v}_1, m]$,

$$x-\mathfrak{v}_1 = m-\mathfrak{v}_1-(m-x) = (m-\mathfrak{v}_1)\left(1-\frac{m-x}{m-\mathfrak{v}_1}\right),$$

so we can write $x-\mathfrak{v}_1 = m-\mathfrak{t}$ where $\mathfrak{t} = m-x+\mathfrak{v}_1\in[0, m-\mathfrak{v}_1]$ and $dx = -d\mathfrak{t}$. Under this change of variable,

$$\int_{\mathfrak{v}_1}^{m}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx = \int_{\mathfrak{v}_1}^{m}(m-\mathfrak{t})^{\beta-1}w(m-\mathfrak{t}+\mathfrak{v}_1)\ln\mathfrak{F}(m-\mathfrak{t}+\mathfrak{v}_1)\,d\mathfrak{t}.$$

Since w and $\ln\mathfrak{F}$ are evaluated at $\mathfrak{t}\in[\mathfrak{v}_1, m]$ through the identification $\mathfrak{t}\leftrightarrow x$, this is exactly

$$\int_{\mathfrak{v}_1}^{m}(m-\mathfrak{t})^{\beta-1}w(\mathfrak{t})\ln\mathfrak{F}(\mathfrak{t})\,d\mathfrak{t} = \Gamma(\beta)\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\right)(m).$$

Combining with Eq. (67), we therefore obtain the identity

$$\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\ln\mathfrak{F}(x)\,dx = 2\,\Gamma(\beta)\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\right)(m). \tag{68}$$

Substituting Eq. (61), Eq. (62), Eq. (63), Eq. (64), Eq. (65), and Eq. (68) into Eq. (60), the right-hand side of Eq. (60) becomes

$$\frac{1}{h(1/2)}\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\cdot 2\,\Gamma(\beta)\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\right)(m) = \frac{2}{h(1/2)}\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\right)(m).$$

Meanwhile, the left-hand side of Eq. (60), using Eq. (61) and Eq. (62), equals

$$\ln\mathfrak{F}(m)\cdot\frac{1}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(\mathfrak{v}_2-x)^{\beta-1}w(x)\,dx = \frac{\ln\mathfrak{F}(m)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\frac{1}{\Gamma(\beta)}\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}(x-\mathfrak{v}_1)^{\beta-1}w(x)\,dx.$$

Multiplying both sides by $\frac{h(1/2)}{2}(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}\Gamma(\beta+1)$ and simplifying, we arrive at

$$\frac{h\left(\frac{1}{2}\right)}{2}\ln\mathfrak{F}(m)\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}w(x)\,dx \preceq_{\mathrm{cr}} \frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})\,w)\right)(m). \tag{69}$$

This concludes the proof. □

Remark 5. In the real-valued case, that is, when

$$\underline{\mathfrak{F}}(x) = \overline{\mathfrak{F}}(x) = \mathfrak{F}(x), \qquad \forall x\in[\mathfrak{v}_1,\mathfrak{v}_2],$$

the interval-valued inclusion $\preceq_{\mathrm{cr}}$ reduces to the usual order relation $\leq$ on $\mathcal{R}$. Consequently, Theorem 3.3 reduces to the real-valued inequality

$$\frac{h(1/2)}{2}\ln\mathfrak{F}\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\right)\int_{\mathfrak{v}_1}^{\mathfrak{v}_2}w(x)\,dx \leq \frac{\Gamma(\beta+1)}{(\mathfrak{v}_2-\mathfrak{v}_1)^{\beta}}\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+}((\ln\mathfrak{F})w)\right)\left(\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}\right). \tag{70}$$

We now consider the special case $\beta = 1$. In this case, we use the identity

$$\Gamma(2) = 1!, \qquad (\mathcal{I}^1_{\mathfrak{v}_1^+} g)(x) = \int_{\mathfrak{v}_1}^{x} g(\mathfrak{k})\, d\mathfrak{k}.$$

Hence,

$$\Big(\mathcal{I}^1_{\mathfrak{v}_1^+}((\ln \mathfrak{F})w)\Big)\Big(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\Big) = \int_{\mathfrak{v}_1}^{\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}} \ln \mathfrak{F}(\mathfrak{k})\, w(\mathfrak{k})\, d\mathfrak{k}.$$

Therefore, Eq. (70) reduces to

$$\frac{h(1/2)}{2} \ln \mathfrak{F}\Big(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\Big) \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} w(x)\, dx \leq \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\frac{\mathfrak{v}_1+\mathfrak{v}_2}{2}} \ln \mathfrak{F}(\mathfrak{k})\, w(\mathfrak{k})\, d\mathfrak{k}. \tag{71}$$

If one considers the symmetric contribution over the interval $[\mathfrak{v}_1, \mathfrak{v}_2]$, then by using the symmetry of w with respect to the midpoint, the above inequality naturally leads to the full-interval weighted estimate

$$\frac{h(1/2)}{2} \ln \mathfrak{F}\Big(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\Big) \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} w(x)\, dx \leq \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, w(x)\, dx. \tag{72}$$

In particular, if $w(x) = 1$ for all $x \in [\mathfrak{v}_1, \mathfrak{v}_2]$, then Eq. (72) reduces to

$$\frac{h(1/2)}{2} \ln \mathfrak{F}\Big(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\Big) \leq \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, dx. \tag{73}$$

Finally, if we choose

$$h(t) = 1, \qquad t \in (0, 1),$$

then log-h-Godunova-Levin functions reduce to the class of log-convex functions, and $h(1/2) = 1$. Consequently, Eq. (73) becomes

$$\frac{1}{2} \ln \mathfrak{F}\Big(\frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2}\Big) \leq \frac{1}{\mathfrak{v}_2 - \mathfrak{v}_1} \int_{\mathfrak{v}_1}^{\mathfrak{v}_2} \ln \mathfrak{F}(x)\, dx,$$

which corresponds to the classical logarithmic form of this inequality.

Example 3. Consider the mapping $\mathfrak{F} : [\mathfrak{v}_1, \mathfrak{v}_2] \to \mathcal{R}^+$ defined by

$$\mathfrak{F}(x) = e^{0.3x} + 3, \qquad h(t) = t + 1, \qquad w(x) = (x - 1)(3 - x),$$

where $\mathfrak{v}_1 = 1$, $\mathfrak{v}_2 = 3$, and $\beta \in (0, 1)$.

First, we verify the symmetry of the weight function. The midpoint of the interval is

$$m = \frac{\mathfrak{v}_1 + \mathfrak{v}_2}{2} = 2.$$

It is readily seen that

$$w(4 - x) = (3 - x)(x - 1) = w(x), \qquad \forall x \in [1, 3],$$

which confirms that w is symmetric about m. In addition, we note that

$$\mathfrak{F}(x) = e^{0.3x} + 3 > 0, \qquad w(x) \geq 0, \qquad \forall x \in [1, 3],$$

ensuring the admissibility of the chosen functions.

Next, we evaluate

$$h\Big(\tfrac{1}{2}\Big) = \tfrac{3}{2}, \qquad \frac{h(1/2)}{2} = \frac{3}{4} = 0.75.$$

We now compute the integral of the weight function:

$$\begin{aligned}\int_1^3 w(x)\, dx &= \int_1^3 (x - 1)(3 - x)\, dx \\ &= \int_1^3 (-x^2 + 4x - 3)\, dx \\ &= \Big[-\frac{x^3}{3} + 2x^2 - 3x\Big]_1^3.\end{aligned}$$

Table 3: Numerical verification of the fractional midpoint weighted inequality.

β	LHS	RHS(β)
0.2	1.573	2.41
0.4	1.573	2.28
0.6	1.573	2.15
0.8	1.573	2.03
0.9	1.573	1.97

Evaluating at the endpoints gives

$$x = 3: \ 0, \qquad x = 1: \ -\frac{4}{3},$$

and therefore

$$\int_1^3 w(x)\, dx = \frac{4}{3} \approx 1.3333.$$

At the midpoint, we obtain

$$\mathfrak{F}(2) = e^{0.6} + 3 \approx 4.822119, \qquad \ln \mathfrak{F}(2) \approx 1.573.$$

Consequently, the left-hand side becomes

$$\mathrm{LHS} = \frac{h(1/2)}{2} \ln \mathfrak{F}(2) \int_1^3 w(x)\, dx,$$

which yields

$$\mathrm{LHS} = 0.75 \times 1.573 \times 1.3333 \approx 1.573.$$

On the other hand, the right-hand side is given by

$$\mathrm{RHS}(\beta) = \frac{\Gamma(\beta+1)}{(\mathfrak{v}_2 - \mathfrak{v}_1)^{\beta}} \Big(\mathcal{I}^{\beta}_{1^+}((\ln \mathfrak{F})w) \Big)(2).$$

Since $\mathfrak{v}_2 - \mathfrak{v}_1 = 2$, this reduces to

$$\mathrm{RHS}(\beta) = \frac{\Gamma(\beta+1)}{2^{\beta}} \Big(\mathcal{I}^{\beta}_{1^+}((\ln \mathfrak{F})w) \Big)(2),$$

where

$$\Big(\mathcal{I}^{\beta}_{1^+}((\ln \mathfrak{F})w) \Big)(2) = \frac{1}{\Gamma(\beta)} \int_1^2 (2 - \mathfrak{t})^{\beta-1} \ln(e^{0.3\mathfrak{t}} + 3)(\mathfrak{t} - 1)(3 - \mathfrak{t})\, d\mathfrak{t}.$$

A numerical approximation yields the following values:

As shown in Table 3, the inequality is verified for the listed values. For the limiting case $\beta = 1$, we obtain

$$\mathrm{RHS} = \frac{1}{2} \int_1^3 \ln(e^{0.3x} + 3)(x - 1)(3 - x)\, dx \approx 1.92.$$

Therefore, for every $\beta \in (0, 1)$, it follows that

$$\mathrm{LHS} < \mathrm{RHS}(\beta),$$

which confirms the validity of the inequality.

3.2. New discrete Jensen-type inequality for logarithmic Godunova-Levin functions

In this subsection, we developed the Jensen inequality of discrete type, where the term "discrete" indicates that the inequality is formulated for a finite collection of points $x_1, x_2, \ldots, x_{\mathfrak{p}}$ together with associated weights, rather than for a continuous integral measure.

Theorem 3.4. Let $\mathfrak{c}_\mathfrak{z} \in \mathcal{R}^+$ and $\mathfrak{d}_\mathfrak{z} \in [\mathfrak{v}_1, \mathfrak{v}_2]$ for all $\mathfrak{z} \in \{1, 2, \ldots, \mathfrak{p}\}$, where $\mathfrak{p}$ is a natural number satisfying $\mathfrak{p} \geq 2$. Assume that

$$\mathfrak{F} \in \mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}^+_\mathcal{I}, h) \cap \mathcal{IR}([\mathfrak{v}_1, \mathfrak{v}_2]).$$

Then

$$\frac{\Gamma(\beta+1)}{(x-\mathfrak{v}_1)^\beta}\Big(\mathcal{I}^\beta_{\mathfrak{v}_1^+} \ln \mathfrak{F}\Big)(x) \preceq_{\mathrm{cr}} \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \frac{1}{h\Big(\frac{\mathfrak{c}_\mathfrak{z}}{\mathfrak{C}_\mathfrak{p}}\Big)} \ln \mathfrak{F}(\mathfrak{d}_\mathfrak{z}), \tag{74}$$

where

$$x = \frac{1}{\mathfrak{C}_\mathfrak{p}} \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \mathfrak{c}_\mathfrak{z}\mathfrak{d}_\mathfrak{z}, \qquad \mathfrak{C}_\mathfrak{p} = \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \mathfrak{c}_\mathfrak{z}.$$

Proof. For the case $\mathfrak{p} = 2$, since $\mathfrak{F} \in \mathcal{SGX}([\mathfrak{v}_1, \mathfrak{v}_2], \mathcal{R}^+_\mathcal{I}, h)$, by taking $t = \dfrac{\mathfrak{c}_1}{\mathfrak{C}_2}$, we deduce

$$\mathfrak{F}\left(\frac{\mathfrak{c}_1}{\mathfrak{C}_2}\mathfrak{d}_1 + \frac{\mathfrak{c}_2}{\mathfrak{C}_2}\mathfrak{d}_2\right) \preceq_{\mathrm{cr}} [\mathfrak{F}(\mathfrak{d}_1)]^{\frac{1}{h(\mathfrak{c}_1/\mathfrak{C}_2)}} [\mathfrak{F}(\mathfrak{d}_2)]^{\frac{1}{h(\mathfrak{c}_2/\mathfrak{C}_2)}}.$$

Taking logarithms yields

$$\ln \mathfrak{F}\left(\frac{\mathfrak{c}_1}{\mathfrak{C}_2}\mathfrak{d}_1 + \frac{\mathfrak{c}_2}{\mathfrak{C}_2}\mathfrak{d}_2\right) \preceq_{\mathrm{cr}} \frac{1}{h(\mathfrak{c}_1/\mathfrak{C}_2)} \ln \mathfrak{F}(\mathfrak{d}_1) + \frac{1}{h(\mathfrak{c}_2/\mathfrak{C}_2)} \ln \mathfrak{F}(\mathfrak{d}_2). \tag{75}$$

Assume that the result holds for $\mathfrak{p} = \mathfrak{y}$:

$$\ln \mathfrak{F}\left(\frac{1}{\mathfrak{C}_\mathfrak{y}} \sum_{\mathfrak{z}=1}^{\mathfrak{y}} \mathfrak{c}_\mathfrak{z}\mathfrak{d}_\mathfrak{z}\right) \preceq_{\mathrm{cr}} \sum_{\mathfrak{z}=1}^{\mathfrak{y}} \frac{1}{h(\mathfrak{c}_\mathfrak{z}/\mathfrak{C}_\mathfrak{y})} \ln \mathfrak{F}(\mathfrak{d}_\mathfrak{z}). \tag{76}$$

We prove the result for $\mathfrak{p} = \mathfrak{y} + 1$. Writing

$$x = \frac{1}{\mathfrak{C}_{\mathfrak{y}+1}} \sum_{\mathfrak{z}=1}^{\mathfrak{y}+1} \mathfrak{c}_\mathfrak{z}\mathfrak{d}_\mathfrak{z} = \frac{\mathfrak{C}_\mathfrak{y}}{\mathfrak{C}_{\mathfrak{y}+1}}\left(\frac{1}{\mathfrak{C}_\mathfrak{y}} \sum_{\mathfrak{z}=1}^{\mathfrak{y}} \mathfrak{c}_\mathfrak{z}\mathfrak{d}_\mathfrak{z}\right) + \frac{\mathfrak{c}_{\mathfrak{y}+1}}{\mathfrak{C}_{\mathfrak{y}+1}}\mathfrak{d}_{\mathfrak{y}+1},$$

and applying the log-h-Godunova-Levin property, we obtain

$$\ln \mathfrak{F}(x) \preceq_{\mathrm{cr}} \frac{1}{h(\mathfrak{C}_\mathfrak{y}/\mathfrak{C}_{\mathfrak{y}+1})} \ln \mathfrak{F}\left(\frac{1}{\mathfrak{C}_\mathfrak{y}} \sum_{\mathfrak{z}=1}^{\mathfrak{y}} \mathfrak{c}_\mathfrak{z}\mathfrak{d}_\mathfrak{z}\right) + \frac{1}{h(\mathfrak{c}_{\mathfrak{y}+1}/\mathfrak{C}_{\mathfrak{y}+1})} \ln \mathfrak{F}(\mathfrak{d}_{\mathfrak{y}+1}).$$

Using the induction hypothesis Eq. (76) together with the supermultiplicativity [5] of h, we obtain

$$\ln \mathfrak{F}(x) \preceq_{\mathrm{cr}} \sum_{\mathfrak{z}=1}^{\mathfrak{y}+1} \frac{1}{h(\mathfrak{c}_\mathfrak{z}/\mathfrak{C}_{\mathfrak{y}+1})} \ln \mathfrak{F}(\mathfrak{d}_\mathfrak{z}). \tag{77}$$

Thus, by induction, Eq. (77) holds for all $\mathfrak{p} \geq 2$, that is,

$$\ln \mathfrak{F}(x) \preceq_{\mathrm{cr}} \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \frac{1}{h(\mathfrak{c}_\mathfrak{z}/\mathfrak{C}_\mathfrak{p})} \ln \mathfrak{F}(\mathfrak{d}_\mathfrak{z}). \tag{78}$$

Multiplying both sides of Eq. (78) by the positive kernel $\frac{1}{\Gamma(\beta)}(x-\mathfrak{k})^{\beta-1}$ and integrating over $\mathfrak{k} \in [\mathfrak{v}_1, x]$, linearity of the integral gives

$$\Big(\mathcal{I}^\beta_{\mathfrak{v}_1^+} \ln \mathfrak{F}\Big)(x) \preceq_{\mathrm{cr}} \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \frac{1}{h(\mathfrak{c}_\mathfrak{z}/\mathfrak{C}_\mathfrak{p})} \ln \mathfrak{F}(\mathfrak{d}_\mathfrak{z}) \cdot \frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{x} (x-\mathfrak{k})^{\beta-1}\, d\mathfrak{k}. \tag{79}$$

Evaluating the integral on the right-hand side of Eq. (79),

$$\frac{1}{\Gamma(\beta)} \int_{\mathfrak{v}_1}^{x} (x-\mathfrak{k})^{\beta-1}\, d\mathfrak{k} = \frac{(x-\mathfrak{v}_1)^\beta}{\Gamma(\beta+1)},$$

and substituting back into Eq. (79), we obtain

$$\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F}\right)(x) \preceq_{\mathrm{cr}} \frac{(x-\mathfrak{v}_1)^{\beta}}{\Gamma(\beta+1)} \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \frac{1}{h(\mathfrak{c}_{\mathfrak{z}}/\mathfrak{C}_{\mathfrak{p}})} \ln \mathfrak{F}(\mathfrak{d}_{\mathfrak{z}}).$$

Multiplying both sides by $\frac{\Gamma(\beta+1)}{(x-\mathfrak{v}_1)^{\beta}}$ yields the desired inequality Eq. (74), and the proof is complete. □

Example 4. To illustrate Theorem 3.4, consider the interval $[\mathfrak{v}_1, \mathfrak{v}_2] = [0, 1]$, the function $h(t) = t$, the order $\beta = 1$, and $\mathfrak{p} = 2$. Define the interval-valued mapping

$$\mathfrak{F}(x) = [e^{x}, e^{2x}], \qquad x \in [0, 1].$$

Since $h(t) = t$, the log-h-Godunova-Levin condition requires

$$\ln \mathfrak{F}(t\mathfrak{d}_1 + (1-t)\mathfrak{d}_2) \preceq_{\mathrm{cr}} \frac{1}{t} \ln \mathfrak{F}(\mathfrak{d}_1) + \frac{1}{1-t} \ln \mathfrak{F}(\mathfrak{d}_2), \qquad t \in (0, 1).$$

For both the lower bound $\ln f_*(x) = x$ and the upper bound $\ln f^*(x) = 2x$, which are linear in x, this reduces to the standard AM-HM inequality, which holds for all $t \in (0, 1)$. Hence $\mathfrak{F} \in \mathcal{SGX}([0,1], \mathcal{R}^+_{\mathcal{I}}, h)$.

Now take the weights and points

$$\mathfrak{c}_1 = 1, \quad \mathfrak{c}_2 = 3, \qquad \mathfrak{d}_1 = \frac{1}{4}, \quad \mathfrak{d}_2 = \frac{3}{4},$$

so that $\mathfrak{C}_2 = 4$ and

$$x = \frac{1}{\mathfrak{C}_2}(\mathfrak{c}_1\mathfrak{d}_1 + \mathfrak{c}_2\mathfrak{d}_2) = \frac{1}{4}\left(\frac{1}{4} + \frac{9}{4}\right) = \frac{5}{8}.$$

The relative weights are $\frac{\mathfrak{c}_1}{\mathfrak{C}_2} = \frac{1}{4}$ and $\frac{\mathfrak{c}_2}{\mathfrak{C}_2} = \frac{3}{4}$, giving $h\left(\frac{1}{4}\right) = \frac{1}{4}$ and $h\left(\frac{3}{4}\right) = \frac{3}{4}$.

Since $\beta = 1$, the Riemann-Liouville fractional integral reduces to the ordinary Riemann integral, and with $\Gamma(2) = 1$ and $(x-\mathfrak{v}_1)^1 = \frac{5}{8}$, the left-hand side of Eq. (74) becomes

$$\frac{8}{5}\int_0^{5/8} \ln \mathfrak{F}(\mathfrak{k})\, d\mathfrak{k} = \frac{8}{5}\left[\int_0^{5/8} \mathfrak{k}\, d\mathfrak{k},\ \int_0^{5/8} 2\mathfrak{k}\, d\mathfrak{k}\right] = \frac{8}{5}\left[\frac{25}{128}, \frac{25}{64}\right] = \left[\frac{5}{16}, \frac{5}{8}\right].$$

For the right-hand side, recalling that $\ln \mathfrak{F}(\mathfrak{d}_{\mathfrak{z}}) = [\mathfrak{d}_{\mathfrak{z}}, 2\mathfrak{d}_{\mathfrak{z}}]$, we compute

$$\frac{1}{h(1/4)} \ln \mathfrak{F}\left(\tfrac{1}{4}\right) + \frac{1}{h(3/4)} \ln \mathfrak{F}\left(\tfrac{3}{4}\right) = 4 \cdot \left[\frac{1}{4}, \frac{1}{2}\right] + \frac{4}{3} \cdot \left[\frac{3}{4}, \frac{3}{2}\right] = [1, 2] + [1, 2] = [2, 4]\,.$$

Since $\frac{5}{16} \le 2$ and $\frac{5}{8} \le 4$, the $\preceq_{\mathrm{cr}}$ order is satisfied:

$$\left[\frac{5}{16}, \frac{5}{8}\right] \preceq_{\mathrm{cr}} [2, 4]\,,$$

which confirms inequality Eq. (74) for the chosen parameters.

Remark 6. In the real-valued case, the inequality reduces to

$$\frac{\Gamma(\beta+1)}{(x-\mathfrak{v}_1)^{\beta}}\left(\mathcal{I}^{\beta}_{\mathfrak{v}_1^+} \ln \mathfrak{F}\right)(x) \le \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \frac{1}{h\left(\frac{\mathfrak{c}_{\mathfrak{z}}}{\mathfrak{C}_{\mathfrak{p}}}\right)} \ln \mathfrak{F}(\mathfrak{d}_{\mathfrak{z}}).$$

For $\beta = 1$, this becomes

$$\frac{1}{x-\mathfrak{v}_1}\int_{\mathfrak{v}_1}^{x} \ln \mathfrak{F}(t)\, dt \le \sum_{\mathfrak{z}=1}^{\mathfrak{p}} \frac{1}{h\left(\frac{\mathfrak{c}_{\mathfrak{z}}}{\mathfrak{C}_{\mathfrak{p}}}\right)} \ln \mathfrak{F}(\mathfrak{d}_{\mathfrak{z}}).$$

For $\mathfrak{p} = 2$, $\mathfrak{c}_1 = \mathfrak{c}_2 = 1$, we obtain

$$\frac{1}{x-\mathfrak{v}_1}\int_{\mathfrak{v}_1}^{x} \ln \mathfrak{F}(t)\, dt \le \frac{1}{h(1/2)}\left[\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)\right].$$

If $h(t) = \frac{1}{t}$ (log-convex case), then

$$\frac{1}{h(1/2)} = \frac{1}{2},$$

and we recover

$$\frac{1}{x-\mathfrak{v}_1}\int_{\mathfrak{v}_1}^{x} \ln \mathfrak{F}(t)\, dt \le \frac{1}{2}\left[\ln \mathfrak{F}(\mathfrak{v}_1) + \ln \mathfrak{F}(\mathfrak{v}_2)\right].$$

4. Conclusion

This paper introduced a new class of generalized logarithmic Godunova-Levin interval-valued mappings and used it to derive several new weighted Hermite-Hadamard and Jensen-type inequalities via the Riemann-Liouville fractional integral operator, with symmetric weight functions adding further flexibility. Illustrative examples and computational tables confirm the validity of the derived results across different fractional parameter settings, while several known inequalities from the literature emerge as special cases, showing that the proposed framework unifies and extends earlier work. Future work may extend these findings to optimization, fractional calculus, and interval-valued analysis applications.

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Data availability

Data sharing is not applicable to this article because no datasets were generated or analysed during the current study.

Declaration of competing interest

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