Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://asr.nsps.org.ng/index.php/asr

African Scientific Reports 5 (2026) 505

African Scientific Reports

On complex fuzzy multisets

Idris Muhammadu Adamu[a,*], Musa Balarabe[b], Abdussamad Tanko Imam[b], Maryam Aliyu[c], Yakubu Nasallah[d]

[a]*Department of Mathematics, Federal University Dutse, P.M.B. 7156 Dutse, Jigawa State, Nigeria*
[b]*Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria*
[c]*Basic Science Units, Maryam Abacha American University-Kano, Nigeria*
[d]*Department of Foundation Courses and Remedial Studies, Binyaminu Usman Polytechnic, Hadejia, Jigawa State, Nigeria*

Abstract

This article proposes a mathematical structure called complex fuzzy multisets (CFMSs), which integrates the multiplicity feature of fuzzy multisets with the complex-valued membership grades of complex fuzzy sets. In this framework, each membership grade is expressed as $re^{i\varphi}$, with $r \in [0, 1]$ and $\varphi \in [0, 2\pi)$, enabling simultaneous representation of membership intensity, phase information, and multiple occurrences of elements. We provide a formal definition of CFMSs and establish fundamental operations, including inclusion, complement, intersection, union, Cartesian product, rotation, and reflection. The algebraic analysis proves that CFMSs satisfy involution, idempotence, commutativity, associativity, distributivity, De Morgan's laws, and absorption, thereby forming a distributive lattice with an involution. Additionally, we define the image and inverse image of CFMSs under mappings. This work lays a rigorous foundation for further theoretical development and potential applications in multi-source data fusion, periodic signal processing, and reasoning under uncertainty with repeated observations.

DOI: 10.46481/asr.2026.5.3.505

Keywords: Fuzzy sets, Complex fuzzy sets, Multisets, Complex fuzzy multisets

Article History:
Received: 01 April 2026
Received in revised form: 31 July 2026
Accepted for publication: 01 August 2026
Available online: 31 August 2026

1. Introduction

Classical set theory, founded by George Cantor, describes a set as a well-defined collection of distinct objects. Although this framework is the foundation of modern mathematics, it cannot directly represent the vagueness or ambiguity that occurs in many real-world problems. The need to handle such imprecision became especially important in fields such as artificial intelligence, pattern recognition, and decision-making.

Zadeh [1] addressed this limitation by introducing fuzzy sets, where each element belongs to a set with a degree of membership in the interval $[0, 1]$. This allowed for gradual, rather than binary, membership. Later, the concept of a multiset [2, 3] permitted an element to appear multiple times, with each occurrence possibly carrying a different membership value. Miyamoto [4] introduced fuzzy multisets by allowing each element to have multiple membership degrees while preserving decreasing order, thereby combining

*Corresponding Author Tel. No.: +234-706-772-0843.
Email address: im.adam@fud.edu.ng (Idris Muhammadu Adamu)

the membership-graded nature of fuzzy sets with the multiplicity feature of multisets. Miyamoto [5] also showed that multisets are a common tool and a fundamental framework in information processing, and reviewed their properties, advanced operations, and applications to rough sets, fuzzy data retrieval, and automatic classification. Further background on fuzzy multisets, algebraic structures on fuzzy sets, and triangular fuzzy multisets are available in Refs. [6–14].

Independently, Ramot *et al*. [15] proposed complex fuzzy sets, extending the membership range from $[0, 1]$ to the unit disk in the complex plane. Each membership value is written as $re^{i\varphi}$, where $r \in [0, 1]$ and $\varphi \in [0, 2\pi)$. This two-dimensional representation simultaneously encodes magnitude, which represents traditional membership strength, and phase, which can represent periodic or directional information. Zhang *et al*. [16] investigated the operational properties and δ-equalities of complex fuzzy sets, introducing a distance measure to quantify differences in both the magnitude and the phase of two complex fuzzy sets. Al-Qudah and Hassan [17] presented complex multi-fuzzy sets in a fixed-dimension paradigm.

The complex fuzzy multisets (CFMSs) proposed in this paper adopt the variable-length fuzzy-multiset paradigm of Miyamoto [4], in which different elements may have different multiplicities. This requires joint-length standardization with default pairs. We also define rotation, reflection, and a distributive lattice with involution; these features do not appear in Ref. [17]. The motivation for introducing this theory arises from real-world scenarios in which information is uncertain, periodic, and may occur in multiple instances. In multi-source data fusion, sensors provide repeated observations with varying confidence; in periodic signal processing, both magnitude and phase are essential; and in decision-making, elements may appear multiple times with different membership strengths. No existing framework in the fuzzy-multiset tradition combines variable-length multiplicity with complex-valued membership grades.

The main contributions of this paper are as follows: (1) a formal definition of CFMSs; (2) fundamental operations, including inclusion, complement, intersection, union, Cartesian product, rotation, and reflection; (3) an algebraic proof that CFMSs form a distributive lattice with an involution; and (4) definitions of image and inverse image under mappings. The paper is organized as follows. Section 2 recalls background on fuzzy sets, multisets, fuzzy multisets, and complex fuzzy sets. Section 3 defines CFMSs. Section 4 introduces set-theoretic operations. Section 5 proves algebraic laws. Section 6 defines image and inverse image. Section 7 concludes with findings, limitations, and future directions.

2. Preliminaries

Definition 2.1 (Fuzzy set)**.** Let X be a nonempty set. A fuzzy set A on the universe X is characterized by the membership function in Eq. (1) [1]:

$$\mu_A : X \longrightarrow [0, 1]. \tag{1}$$

The value $\mu_A(x)$ describes the degree to which x belongs to A.

Definition 2.2 (Multiset)**.** Let X be a set. A multiset A over X is a pair $\langle X, C_A\rangle$, where the count function in Eq. (2) assigns to each element its non-negative integer multiplicity [3]:

$$C_A : X \longrightarrow \mathbb{N}_0. \tag{2}$$

An ordinary set corresponds to the case in which C_A takes only the values 0 and 1. The collection of all multisets over X is denoted by $MS(X)$.

Definition 2.3 (Fuzzy multiset)**.** Let X be a nonempty set. A fuzzy multiset (FMS) A drawn from X is characterized by the count-membership function in Eq. (3) [4]:

$$CM_A : X \longrightarrow Q, \tag{3}$$

where Q is the set of all crisp multisets drawn from the unit interval $[0, 1]$. For any $x \in X$, $CM_A(x)$ is a crisp multiset drawn from $[0, 1]$. For each $x \in X$, the membership sequence is the decreasingly ordered sequence

$$(\mu_1^A(x), \mu_2^A(x), \ldots, \mu_p^A(x)), \quad \mu_1^A(x) \geq \mu_2^A(x) \geq \cdots \geq \mu_p^A(x).$$

Definition 2.4 (Complex fuzzy set)**.** A complex fuzzy set (CFS) A, defined on a universe of discourse U, is characterized by a membership function that assigns to each element $x \in U$ a complex-valued membership grade in the unit circle [15]. The membership grade is given in Eq. (4):

$$\mu_A(x) = r_A(x)e^{i\varphi_A(x)}, \qquad r_A(x) \in [0, 1]. \tag{4}$$

The CFS A may be represented as

$$A = \{\langle x, \mu_A(x)\rangle : x \in U\}.$$

3. Introducing complex fuzzy multisets

Definition 3.1 (Complex fuzzy multiset)**.** Let M be a nonempty crisp set. A complex fuzzy multiset (CFMS) H on M is characterized by the count complex membership function in Eq. (5):

$$CCM_H : M \longrightarrow Q_C, \tag{5}$$

where Q_C is the collection of all crisp multisets drawn from the complex unit disk

$$\{z \in \mathbb{C} : |z| \leq 1\}.$$

For each $g \in M$, the membership sequence is a decreasingly ordered sequence, by magnitude, of complex numbers from $CCM_H(g)$:

$$(\mu_H^1(g), \mu_H^2(g), \dots, \mu_H^{m_g}(g)), \quad |\mu_H^1(g)| \geq |\mu_H^2(g)| \geq \cdots \geq |\mu_H^{m_g}(g)|.$$

Each complex membership grade is expressed in polar form as in Eq. (6):

$$\mu_H^k(g) = \eta_H^k(g) e^{i\varphi_H^k(g)}, \quad \eta_H^k(g) \in [0,1], \quad \varphi_H^k(g) \in [0, 2\pi). \tag{6}$$

Here $\eta_H^k(g)$ represents amplitude, or membership intensity, and $\varphi_H^k(g)$ represents phase angle, or periodic/directional information.

A CFMS H is completely described by

$$H = \left\{ \left\langle g, (\mu_H^1(g), \mu_H^2(g), \dots, \mu_H^{m_g}(g)) \right\rangle : g \in M \right\},$$

or equivalently,

$$H = \{\langle g, \mu_H^k(g) \rangle : g \in M,\ k = 1, 2, \dots, m_g\}.$$

The magnitudes $\eta_H^k(g)$ are taken from $[0,1]$ to maintain the standard fuzzy membership scale, ensuring that each complex membership grade lies within the unit disk. The phase angles $\varphi_H^k(g)$ are taken from $[0, 2\pi)$ because this is the conventional interval for representing a full cycle of periodic information without ambiguity.

Example 1. Let $G = \{a, b, c\}$. Define a CFMS H on G by

$$H = \{\langle a, (0.8e^{i\pi/4}, 0.6e^{i\pi/6}, 0.4e^{i\pi/3}) \rangle, \\ \langle b, (0.9e^{i\pi/2}, 0.7e^{i\pi/3}) \rangle, \langle c, (0.5e^{i\pi/2}) \rangle\}.$$

For a, $0.8 \geq 0.6 \geq 0.4$; for b, $0.9 \geq 0.7$; and for c, the single membership grade has magnitude 0.5. Hence the membership sequences satisfy the decreasing-magnitude condition.

Definition 3.2 (Length)**.** Let H be a CFMS on M. The length of an element $g \in M$ in H is the size of the count complex membership multiset $CCM_H(g)$. The length is given in Eq. (7):

$$L(g : H) = |CCM_H(g)| = m_g^H, \tag{7}$$

where m_g^H is the multiplicity of g in H.

Definition 3.3 (Joint length)**.** Let H_1 and H_2 be two CFMSs drawn from M. The joint length of an element $g \in M$ for H_1 and H_2 is defined in Eq. (8):

$$L(g : H_1, H_2) = \max\{L(g : H_1), L(g : H_2)\} = \max\{m_g^{H_1}, m_g^{H_2}\}. \tag{8}$$

When the CFMSs H_1 and H_2 are clear from context, we write $L(g)$.

Example 2. Let $M = \{g_1, g_2, g_3, g_4\}$. Consider

$$H_1 = \{\langle g_1, 0.3e^{i\pi/6} \rangle, \langle g_1, 0.2e^{i\pi/3} \rangle, \langle g_2, 1.0e^{i0} \rangle, \langle g_2, 0.5e^{i\pi/4} \rangle, \\ \langle g_3, 0.5e^{i\pi/3} \rangle, \langle g_3, 0.5e^{i\pi/6} \rangle, \langle g_3, 0.4e^{i\pi/4} \rangle, \langle g_3, 0.3e^{i\pi/3} \rangle, \langle g_3, 0.2e^{i\pi/2} \rangle\},$$

and

$$H_2 = \{\langle g_1, 0.4e^{i\pi/4} \rangle, \langle g_2, 1.0e^{i0} \rangle, \langle g_2, 0.3e^{i\pi/6} \rangle, \langle g_2, 0.2e^{i\pi/3} \rangle, \langle g_4, 0.2e^{i\pi/4} \rangle, \langle g_4, 0.1e^{i\pi/6} \rangle\}.$$

The lengths are

$$L(g_1 : H_1) = 2, \quad L(g_2 : H_1) = 2, \quad L(g_3 : H_1) = 5, \quad L(g_4 : H_1) = 0, \\ L(g_1 : H_2) = 1, \quad L(g_2 : H_2) = 3, \quad L(g_3 : H_2) = 0, \quad L(g_4 : H_2) = 2.$$

For operations such as inclusion, intersection, and union, two CFMSs must have the same length for each element $g \in M$. If the lengths differ, the shorter multiset is standardized by appending the default pair

$$\langle g, 0.0e^{i0} \rangle,$$

which represents zero membership. Since the magnitude is zero, the choice of phase is arbitrary and does not affect the results of union, intersection, or inclusion operations. The collection of all CFMSs defined over M is denoted by

$$\text{CFMS}(M) = \{H : H \text{ is a CFMS on } M\}.$$

Table 1: Componentwise inclusion check for p.

j	$\eta^j_{H_1}(p) \leq \eta^j_{H_2}(p)$	$\varphi^j_{H_1}(p) \leq \varphi^j_{H_2}(p)$	Result
1	$0.4 \leq 0.6$	$\pi/6 \leq \pi/3$	True
2	$0.3 \leq 0.5$	$\pi/4 \leq \pi/2$	True
3	$0 \leq 0.4$	$0 \leq \pi$	True

Table 2: Componentwise inclusion check for q.

j	$\eta^j_{H_1}(q) \leq \eta^j_{H_2}(q)$	$\varphi^j_{H_1}(q) \leq \varphi^j_{H_2}(q)$	Result
1	$0.5 \leq 0.7$	$\pi/3 \leq \pi/2$	True
2	$0 \leq 0.6$	$0 \leq 2\pi/3$	True

Table 3: Summary of CFMS inclusion and complement operations.

Operation	Condition/definition	Example
Inclusion $H_1 \subseteq H_2$	$\eta^j_{H_1}(g) \leq \eta^j_{H_2}(g)$ and $\varphi^j_{H_1}(g) \leq \varphi^j_{H_2}(g)$ for all g, j	$H_1 \subseteq H_2$ is true in Tables 1 and 2
Complement H^c	$\mu^k_{H^c}(g) = (1 - \eta^k_H(g))e^{i(2\pi - \varphi^k_H(g))}$	$0.8e^{i\pi/4} \mapsto 0.2e^{i7\pi/4}$

4. Set-theoretic operations on complex fuzzy multisets

Definition 4.1 (Inclusion)**.** Let H_1 and H_2 be two CFMSs on M. Then H_1 is contained in H_2, denoted $H_1 \subseteq H_2$, if and only if for every $g \in M$ and for each multiplicity index $j = 1, 2, \ldots, L(g)$, the componentwise inequalities in Eq. (9) hold:

$$\eta^j_{H_1}(g) \leq \eta^j_{H_2}(g), \qquad \varphi^j_{H_1}(g) \leq \varphi^j_{H_2}(g). \tag{9}$$

Missing pairs are treated with default values $\eta = 0$ and $\varphi = 0$. Two CFMSs H_1 and H_2 are equal if and only if $H_1 \subseteq H_2$ and $H_2 \subseteq H_1$.

Example 3. Let $G = \{p, q\}$,

$$H_1 = \{\langle p, 0.4e^{i\pi/6}\rangle, \langle p, 0.3e^{i\pi/4}\rangle, \langle q, 0.5e^{i\pi/3}\rangle\},$$

and

$$H_2 = \{\langle p, 0.6e^{i\pi/3}\rangle, \langle p, 0.5e^{i\pi/2}\rangle, \langle p, 0.4e^{i\pi}\rangle, \langle q, 0.7e^{i\pi/2}\rangle, \langle q, 0.6e^{i2\pi/3}\rangle\}.$$

The joint lengths are $L(p) = 3$ and $L(q) = 2$. Tables 1 and 2 show that all componentwise inequalities hold; therefore, $H_1 \subseteq H_2$.

Definition 4.2 (Complement)**.** Let

$$H = \{\langle g, \mu^k_H(g)\rangle : g \in M,\ k = 1, 2, \ldots, m_g\}$$

be a CFMS, where $\mu^k_H(g) = \eta^k_H(g)e^{i\varphi^k_H(g)}$, with $\eta^k_H(g) \in [0, 1]$ and $\varphi^k_H(g) \in [0, 2\pi)$. The complement of H, denoted H^c, is defined for each $g \in M$ and each multiplicity index k by Eq. (10):

$$\mu^k_{H^c}(g) = \left(1 - \eta^k_H(g)\right) e^{i(2\pi - \varphi^k_H(g))}. \tag{10}$$

For indices k beyond the multiplicity m^H_g, we take $\eta^k_H(g) = 0$ and $\varphi^k_H(g) = 0$, yielding $\mu^k_{H^c}(g) = 1 \cdot e^{i2\pi} = 1$.

The conjugate form $\eta e^{-i\varphi}$ is reserved for the reflection operation in Definition 4.7. For complement, we adopt the magnitude-negation form in Eq. (10) because it satisfies De Morgan's laws with the min/max definitions of union and intersection.

Example 4. Let

$$H = \{\langle g_1, 0.8e^{i\pi/4}\rangle, \langle g_1, 0.6e^{i\pi/3}\rangle, \langle g_2, 0.7e^{i\pi/2}\rangle, \langle g_2, 0.5e^{i\pi/6}\rangle\}.$$

Using Eq. (10),

$$H^c = \{\langle g_1, 0.2e^{i7\pi/4}\rangle, \langle g_1, 0.4e^{i5\pi/3}\rangle, \langle g_2, 0.3e^{i3\pi/2}\rangle, \langle g_2, 0.5e^{i11\pi/6}\rangle\}.$$

Table 3 summarizes the inclusion and complement operations.

Definition 4.3 (Intersection)**.** Let H_1 and H_2 be two CFMSs on M. The intersection $H_1 \cap H_2$ is defined componentwise by Eq. (11):

$$\mu^k_{H_1 \cap H_2}(g) = \min\{\eta^k_{H_1}(g), \eta^k_{H_2}(g)\}e^{i\min\{\varphi^k_{H_1}(g), \varphi^k_{H_2}(g)\}}. \tag{11}$$

For indices beyond the multiplicity of an element, we take the corresponding missing amplitude and phase to be 0.

Definition 4.4 (Union)**.** Let H_1 and H_2 be two CFMSs on M. The union $H_1 \cup H_2$ is defined componentwise by Eq. (12):

$$\mu^k_{H_1\cup H_2}(g) = \max\{\eta^k_{H_1}(g), \eta^k_{H_2}(g)\}e^{i\max\{\varphi^k_{H_1}(g),\varphi^k_{H_2}(g)\}}. \tag{12}$$

For indices beyond the multiplicity of an element, the same zero-membership convention used for intersection applies.

Definition 4.5 (Cartesian product)**.** Let H_1 and H_2 be two CFMSs on a domain U. The complex fuzzy Cartesian product $H_1 \times H_2$ is defined on $U \times U$ by Eq. (13):

$$\begin{aligned}\mu^k_{H_1\times H_2}(g_1, g_2) &= \eta^k_{H_1\times H_2}(g_1, g_2)e^{i\varphi^k_{H_1\times H_2}(g_1,g_2)}\\ &= \min\{\eta^k_{H_1}(g_1), \eta^k_{H_2}(g_2)\}e^{i\min\{\varphi^k_{H_1}(g_1),\varphi^k_{H_2}(g_2)\}}.\end{aligned} \tag{13}$$

Here $k = 1, 2, \ldots, \min(m^{H_1}_{g_1}, m^{H_2}_{g_2})$. The use of the minimum multiplicity reflects that the k-th membership of (g_1, g_2) is defined only when both H_1 has a k-th membership for g_1 and H_2 has a k-th membership for g_2.

Example 5. Let $M = \{g_1, g_2\}$,

$$H_1 = \{\langle g_1, 0.8e^{i\pi/4}\rangle, \langle g_1, 0.6e^{i\pi/3}\rangle, \langle g_2, 0.7e^{i\pi/2}\rangle, \langle g_2, 0.5e^{i\pi/6}\rangle\},$$

and

$$H_2 = \{\langle g_1, 0.9e^{i\pi/6}\rangle, \langle g_1, 0.4e^{i\pi/4}\rangle, \langle g_1, 0.3e^{i\pi/3}\rangle, \langle g_2, 0.6e^{i\pi/4}\rangle\}.$$

By Eqs. (11)–(13), the operations are

$$\begin{aligned}H_1 \cap H_2 &= \{\langle g_1, 0.8e^{i\pi/6}\rangle, \langle g_1, 0.4e^{i\pi/4}\rangle, \langle g_1, 0\rangle, \langle g_2, 0.6e^{i\pi/4}\rangle, \langle g_2, 0\rangle\},\\ H_1 \cup H_2 &= \{\langle g_1, 0.9e^{i\pi/4}\rangle, \langle g_1, 0.6e^{i\pi/3}\rangle, \langle g_1, 0.3e^{i\pi/3}\rangle, \langle g_2, 0.7e^{i\pi/2}\rangle, \langle g_2, 0.5e^{i\pi/6}\rangle\}.\end{aligned}$$

Definition 4.6 (Rotation)**.** Let

$$H = \{\langle g, \mu^k_H(g)\rangle : g \in M,\ k = 1, 2, \ldots, m_g\}$$

be a CFMS, where $\mu^k_H(g) = \eta^k_H(g)e^{i\varphi^k_H(g)}$. The rotation of H by an angle $\theta \in [0, 2\pi)$ is defined by Eq. (14):

$$R_\theta(H) = \{\langle g, \eta^k_H(g)e^{i(\varphi^k_H(g)+\theta)}\rangle : g \in M,\ k = 1, 2, \ldots, m_g\}. \tag{14}$$

The rotation operation in Eq. (14) satisfies identity, additivity, and periodicity. It is also related to complement, union, and intersection through the properties stated in Section 5.

Definition 4.7 (Reflection)**.** Let $H = \{\langle g, \mu^k_H(g)\rangle : g \in M,\ k = 1, 2, \ldots, m_g\}$ be a CFMS, where $\mu^k_H(g) = \eta^k_H(g)e^{i\varphi^k_H(g)}$. The reflection of H, given by Eq. (15), is

$$\mathrm{Ref}(H) = \{\langle g, \eta^k_H(g)e^{-i\varphi^k_H(g)}\rangle : g \in M,\ k = 1, 2, \ldots, m_g\}. \tag{15}$$

This operation corresponds to taking the complex conjugate of each membership grade.

Example 6. For

$$H = \{\langle g_1, 0.8e^{i\pi/4}\rangle, \langle g_2, 0.6e^{i\pi/3}\rangle\},$$

Eqs. (14) and (15) give

$$R_{\pi/2}(H) = \{\langle g_1, 0.8e^{i3\pi/4}\rangle, \langle g_2, 0.6e^{i5\pi/6}\rangle\},$$

and

$$\mathrm{Ref}(H) = \{\langle g_1, 0.8e^{-i\pi/4}\rangle, \langle g_2, 0.6e^{-i\pi/3}\rangle\}.$$

5. Laws of algebra for complex fuzzy multisets

This section establishes the fundamental algebraic laws of CFMSs. All algebraic properties are established under the convention that phases are ordered linearly on $[0, 2\pi)$, and rotation results are understood modulo 2π.

5.1. Core algebraic laws

Proposition 5.1 (Involution)**.** For $H \in \mathrm{CFMS}(M)$, $(H^c)^c = H$.

Proof. For each $g \in M$ and each index k, Eq. (10) gives

$$\mu^k_{H^c}(g) = \left(1 - \eta^k_H(g)\right) e^{i(2\pi - \varphi^k_H(g))}.$$

Applying the same complement again yields

$$\mu^k_{(H^c)^c}(g) = \left(1 - (1 - \eta^k_H(g))\right) e^{i(2\pi-(2\pi-\varphi^k_H(g)))} = \eta^k_H(g) e^{i\varphi^k_H(g)} = \mu^k_H(g).$$

Hence $(H^c)^c = H$. □

Proposition 5.2 (Idempotence)**.** For $H \in \mathrm{CFMS}(M)$, $H \cup H = H$ and $H \cap H = H$.

Proof. For each $g \in M$ and each $k = 1, 2, \ldots, m^H_g$, Eqs. (12) and (11) give

$$\mu^k_{H\cup H}(g) = \max\{\eta^k_H(g), \eta^k_H(g)\} e^{i \max\{\varphi^k_H(g), \varphi^k_H(g)\}} = \mu^k_H(g),$$

and

$$\mu^k_{H\cap H}(g) = \min\{\eta^k_H(g), \eta^k_H(g)\} e^{i \min\{\varphi^k_H(g), \varphi^k_H(g)\}} = \mu^k_H(g).$$

Therefore $H \cup H = H$ and $H \cap H = H$. □

Proposition 5.3 (Commutativity)**.** For $H_1, H_2 \in \mathrm{CFMS}(M)$, $H_1 \cup H_2 = H_2 \cup H_1$ and $H_1 \cap H_2 = H_2 \cap H_1$.

Proof. The result follows componentwise from the commutativity of max and min on the amplitude and phase components in Eqs. (12) and (11). □

Proposition 5.4 (Associativity)**.** For $H_1, H_2, H_3 \in \mathrm{CFMS}(M)$,

$$(H_1 \cup H_2) \cup H_3 = H_1 \cup (H_2 \cup H_3), \qquad (H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3).$$

Proof. For each $g \in M$ and each admissible k, the amplitude part of the union is

$$\max\{\max\{\eta^k_{H_1}(g), \eta^k_{H_2}(g)\}, \eta^k_{H_3}(g)\} = \max\{\eta^k_{H_1}(g), \max\{\eta^k_{H_2}(g), \eta^k_{H_3}(g)\}\},$$

and the same identity holds for the phase part. Thus union is associative. The proof for intersection is obtained by replacing max with min. □

Proposition 5.5 (Distributivity)**.** For $H_1, H_2, H_3 \in \mathrm{CFMS}(M)$,

$$H_1 \cup (H_2 \cap H_3) = (H_1 \cup H_2) \cap (H_1 \cup H_3),$$

and

$$H_1 \cap (H_2 \cup H_3) = (H_1 \cap H_2) \cup (H_1 \cap H_3).$$

Proof. The proof is componentwise. For amplitudes, the identities

$$\max\{a, \min\{b, c\}\} = \min\{\max\{a, b\}, \max\{a, c\}\}$$

and

$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\}$$

hold for $a, b, c \in [0, 1]$. The same identities hold for the linearly ordered phase components in $[0, 2\pi)$. Applying these identities to Eqs. (11) and (12) gives the stated distributive laws. □

Proposition 5.6 (De Morgan's laws)**.** For $H_1, H_2 \in \mathrm{CFMS}(M)$,

$$(H_1 \cup H_2)^c = H_1^c \cap H_2^c, \qquad (H_1 \cap H_2)^c = H_1^c \cup H_2^c.$$

Table 4: Summary of core algebraic laws for complex fuzzy multisets.

Proposition	Law
5.1	$(H^c)^c = H$ (involution)
5.2	$H \cup H = H$ and $H \cap H = H$ (idempotence)
5.3	$H_1 \cup H_2 = H_2 \cup H_1$ and $H_1 \cap H_2 = H_2 \cap H_1$ (commutativity)
5.4	$(H_1 \cup H_2) \cup H_3 = H_1 \cup (H_2 \cup H_3)$ and $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$ (associativity)
5.5	$H_1 \cup (H_2 \cap H_3) = (H_1 \cup H_2) \cap (H_1 \cup H_3)$ and $H_1 \cap (H_2 \cup H_3) = (H_1 \cap H_2) \cup (H_1 \cap H_3)$ (distributivity)
5.6	$(H_1 \cup H_2)^c = H_1^c \cap H_2^c$ and $(H_1 \cap H_2)^c = H_1^c \cup H_2^c$ (De Morgan's laws)
5.7	$H_1 \cap (H_1 \cup H_2) = H_1$ and $H_1 \cup (H_1 \cap H_2) = H_1$ (absorption)

Proof. For the first identity, Eqs. (12) and (10) give

$$\mu^k_{(H_1 \cup H_2)^c}(g) = \left(1 - \max\{\eta^k_{H_1}(g), \eta^k_{H_2}(g)\}\right) e^{i(2\pi - \max\{\varphi^k_{H_1}(g), \varphi^k_{H_2}(g)\})}.$$

On the other hand, using Eq. (11) on H_1^c and H_2^c gives

$$\mu^k_{H_1^c \cap H_2^c}(g) = \min\{1 - \eta^k_{H_1}(g), 1 - \eta^k_{H_2}(g)\} e^{i \min\{2\pi - \varphi^k_{H_1}(g), 2\pi - \varphi^k_{H_2}(g)\}}.$$

Since $\min\{1-a, 1-b\} = 1 - \max\{a, b\}$ and $\min\{2\pi - \phi_1, 2\pi - \phi_2\} = 2\pi - \max\{\phi_1, \phi_2\}$, the two expressions are equal. The second identity follows by swapping min and max. □

Proposition 5.7 (Absorption)**.** For $H_1, H_2 \in \mathrm{CFMS}(M)$,

$$H_1 \cap (H_1 \cup H_2) = H_1, \qquad H_1 \cup (H_1 \cap H_2) = H_1.$$

Proof. The amplitude and phase components are governed by the absorption laws

$$\min\{a, \max\{a, b\}\} = a, \qquad \max\{a, \min\{a, b\}\} = a.$$

Applying these identities componentwise to Eqs. (11) and (12) proves both absorption identities. □

Table 4 summarizes the core algebraic laws for CFMSs.

5.2. Properties of set rotation and set reflection

Proposition 5.8. For any $H \in \mathrm{CFMS}(M)$ and any $\theta_1, \theta_2 \in [0, 2\pi)$,

$$R_{\theta_1}(R_{\theta_2}(H)) = R_{\theta_1 + \theta_2 \pmod{2\pi}}(H).$$

Proof. For each $g \in M$ and each index k, Eq. (14) gives

$$R_{\theta_1}(R_{\theta_2}(H)) = \{\langle g, \eta^k_H(g) e^{i(\varphi^k_H(g) + \theta_2 + \theta_1)} \rangle\} = R_{\theta_1 + \theta_2}(H).$$

Since $e^{i(\varphi + 2\pi)} = e^{i\varphi}$, the result holds modulo 2π. □

Proposition 5.9. For any $H \in \mathrm{CFMS}(M)$, $\mathrm{Ref}(\mathrm{Ref}(H)) = H$.

Proof. For each $g \in M$ and each index k, Eq. (15) gives

$$\mathrm{Ref}(\mathrm{Ref}(H)) = \{\langle g, \eta^k_H(g) e^{-i(-\varphi^k_H(g))} \rangle\} = \{\langle g, \eta^k_H(g) e^{i\varphi^k_H(g)} \rangle\} = H.$$

□

Proposition 5.10. Let $H_1, H_2 \in \mathrm{CFMS}(M)$ and $\theta \in [0, 2\pi)$. Then

$$R_\theta(H_1 \cup H_2) \subseteq R_\theta(H_1) \cup R_\theta(H_2),$$

$$R_\theta(H_1 \cap H_2) \supseteq R_\theta(H_1) \cap R_\theta(H_2),$$

and

$$R_\theta(H^c) = (R_{-\theta}(H))^c.$$

Proof. The amplitude components are preserved by rotation. The phase components may be affected by modulo 2π wrap-around, so equality with union or intersection need not hold in general. The complement identity follows directly from

$$R_\theta(H^c) = \{\langle g, (1-\eta_H^k(g))e^{i(2\pi-\varphi_H^k(g)+\theta)}\rangle\} = (R_{-\theta}(H))^c.$$

□

Proposition 5.11. Let $H_1, H_2 \in \mathrm{CFMS}(M)$. Then

$$\mathrm{Ref}(H_1 \cup H_2) \subseteq \mathrm{Ref}(H_1) \cup \mathrm{Ref}(H_2),$$

$$\mathrm{Ref}(H_1 \cap H_2) \supseteq \mathrm{Ref}(H_1) \cap \mathrm{Ref}(H_2),$$

and

$$\mathrm{Ref}(H^c) = (\mathrm{Ref}(H))^c.$$

Proof. Reflection preserves amplitudes and reverses phase direction. Thus equality with union and intersection may fail when the phase order is reversed. For the complement identity, Eq. (10) gives

$$\mathrm{Ref}(H^c) = \{\langle g, (1-\eta_H^k(g))e^{-i(2\pi-\varphi_H^k(g))}\rangle\} = \{\langle g, (1-\eta_H^k(g))e^{i\varphi_H^k(g)}\rangle\} = (\mathrm{Ref}(H))^c.$$

□

Proposition 5.12. For a CFMS $H \in \mathrm{CFMS}(M)$ and a fixed $\theta \in [0, 2\pi)$, the following are equivalent:

1. $R_\theta(H) = H$;
2. for each $g \in M$, the multiset of pairs
$$\{(\eta_H^k(g), \varphi_H^k(g)) : k = 1, 2, \ldots, m_g\}$$
is invariant under the map
$$(\eta, \varphi) \mapsto (\eta, \varphi + \theta \pmod{2\pi}).$$

Proof. If $R_\theta(H) = H$, then for each grade $\eta e^{i\varphi}$, its image $\eta e^{i(\varphi+\theta)}$ must also appear in H with the same magnitude. Conversely, if the multiset of grades is closed under addition of θ with equal magnitudes, then rotation by θ merely permutes the grades within each orbit, leaving H unchanged. □

The only CFMS invariant under all rotations is the zero multiset. However, for a fixed θ, nonzero invariant CFMSs exist when the phases are closed under addition of θ with matching magnitudes. In both Eq. (14) and Eq. (15), the magnitude remains unchanged:

$$\eta_{R_\theta(H)}^k(g) = \eta_H^k(g), \qquad \eta_{\mathrm{Ref}(H)}^k(g) = \eta_H^k(g).$$

6. Image and inverse image of complex fuzzy multisets

Definition 6.1 (Image)**.** Let M_1 and M_2 be two nonempty sets, and let $f : M_1 \to M_2$ be a mapping. The image of a CFMS $R \in \mathrm{CFMS}(M_1)$ under f, denoted $f(R) \in \mathrm{CFMS}(M_2)$, is defined for each $g_2 \in M_2$ and each multiplicity index k by Eq. (16):

$$\mu_{f(R)}^k(g_2) = \begin{cases} \sup\limits_{g_1 \in f^{-1}(g_2)} \mu_R^k(g_1), & f^{-1}(g_2) \neq \varnothing, \\ 0 \cdot e^{i0}, & \text{otherwise.} \end{cases} \tag{16}$$

The supremum is computed componentwise. For the magnitude it is the usual supremum in $[0, 1]$, and for the phase it is the least upper bound in the linear order on $[0, 2\pi)$. The multiplicity for each $g_2 \in M_2$ is

$$m_{g_2}^{f(R)} = \begin{cases} \sup\{m_{g_1}^R : g_1 \in f^{-1}(g_2)\}, & f^{-1}(g_2) \neq \varnothing, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 6.2 (Inverse image)**.** Let M_1 and M_2 be two nonempty sets, and let $f : M_1 \to M_2$ be a mapping. The inverse image of a CFMS $S \in \mathrm{CFMS}(M_2)$, denoted $f^{-1}(S) \in \mathrm{CFMS}(M_1)$, is defined for each $g_1 \in M_1$ and each multiplicity index k by Eq. (17):

$$\mu_{f^{-1}(S)}^k(g_1) = \mu_S^k(f(g_1)) = \eta_S^k(f(g_1))e^{i\varphi_S^k(f(g_1))}. \tag{17}$$

The multiplicity is preserved by

$$m_{g_1}^{f^{-1}(S)} = m_{f(g_1)}^S.$$

Example 7. Let $M_1 = \{g_1, g_2\}$, $M_2 = \{h_1, h_2\}$, $f(g_1) = h_1$, and $f(g_2) = h_2$. If

$$R = \{\langle g_1, 0.7e^{i0.4\pi}\rangle, \langle g_1, 0.5e^{i0.3\pi}\rangle, \langle g_2, 0.8e^{i0.5\pi}\rangle\},$$

then Eq. (16) gives

$$f(R) = \{\langle h_1, 0.7e^{i0.4\pi}\rangle, \langle h_1, 0.5e^{i0.3\pi}\rangle, \langle h_2, 0.8e^{i0.5\pi}\rangle\}.$$

If

$$S = \{\langle h_1, 0.9e^{i0.8\pi}\rangle, \langle h_2, 0.6e^{i0.3\pi}\rangle, \langle h_2, 0.4e^{i0.2\pi}\rangle\},$$

then Eq. (17) gives

$$f^{-1}(S) = \{\langle g_1, 0.9e^{i0.8\pi}\rangle, \langle g_2, 0.6e^{i0.3\pi}\rangle, \langle g_2, 0.4e^{i0.2\pi}\rangle\}.$$

7. Conclusion

We defined CFMSs with variable-length complex membership sequences ordered by decreasing magnitude. We proved that CFMSs satisfy involution, idempotence, commutativity, associativity, distributivity, De Morgan's laws, and absorption, thereby forming a distributive lattice with an involution. We also defined the image and inverse image of CFMSs under mappings.

The present study employs the standard min/max operations to establish the lattice structure, while other t-norms and t-conorms remain to be investigated for broader applicability. Although the mathematical foundation is complete, empirical validation in real-world settings is a natural direction for future work. These considerations do not compromise the theoretical correctness of the proposed framework but rather highlight meaningful avenues for further development.

This work opens several future research directions, including developing distance measures and similarity metrics for CFMSs, investigating alternative complement operations and their algebraic consequences, applying CFMSs to multi-source data fusion and periodic signal processing, extending the framework to interval-valued and type-2 CFMSs, developing computational algorithms for efficient CFMS manipulation, and exploring connections to picture fuzzy multisets and neutrosophic multisets.

Data availability

No new data were generated.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

Funding

The authors received no external funding for this study.

References

[1] L. A. Zadeh, "Fuzzy sets", Information and Control **8** (1965) 338. https://doi.org/10.1016/S0019-9958(65)90241-X.
[2] W. D. Blizard, "Multiset theory", Notre Dame Journal of Formal Logic **30** (1989) 36. https://doi.org/10.1305/ndjfl/1093634995.
[3] A. Syropoulos, *Mathematics of multisets*, in *Multiset Processing: WMC 2000*, C. S. Calude, G. Păun, G. Rozenberg & A. Salomaa (Eds.), Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2001, pp. 347–358. https://doi.org/10.1007/3-540-45523-X_17.
[4] S. Miyamoto, *Fuzzy multisets and their generalizations*, in *Multiset Processing: WMC 2000*, C. S. Calude, G. Păun, G. Rozenberg & A. Salomaa (Eds.), Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2001, pp. 225–235. https://doi.org/10.1007/3-540-45523-X_11.
[5] S. Miyamoto, *Multisets and fuzzy multisets as a framework of information systems*, in *Modeling Decisions for Artificial Intelligence*, V. Torra & Y. Narukawa (Eds.), Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2004, pp. 9–18. https://doi.org/10.1007/978-3-540-27774-3_4.
[6] S. Miyamoto, "Remarks on basics of fuzzy sets and fuzzy multisets", Fuzzy Sets and Systems **156** (2005) 427. https://doi.org/10.1016/j.fss.2005.05.040.
[7] S. Sebastian & T. V. Ramakrishnan, "Multi-fuzzy sets: an extension of fuzzy sets", Fuzzy Information and Engineering **3** (2011) 35. https://doi.org/10.1007/s12543-011-0064-y.
[8] D. Singh, A. Alkali & A. M. Ibrahim, "An outline of the development of the concept of fuzzy multisets", International Journal of Innovation, Management and Technology **4** (2013) 209. https://doi.org/10.7763/IJIMT.2013.V4.393.
[9] J. Paul & S. J. John, "Type 2 fuzzy multisets and its applications in decision making problems", Journal of Intelligent and Fuzzy Systems: Applications in Engineering and Technology **30** (2015) 359. https://doi.org/10.3233/IFS-151761.
[10] A. Riesgo, P. Alonso, I. Díaz & S. Montes, "Basic operations for fuzzy multisets", International Journal of Approximate Reasoning **101** (2018) 107. https://doi.org/10.1016/j.ijar.2018.06.008.
[11] J. M. Agbetayo & P. A. Ejegwa, "Certain results on metric and norm in fuzzy multiset setting", Ratio Mathematica **40** (2021) 67. https://doi.org/10.23755/rm.v40i1.582.
[12] A. Syropoulos, "On triangular multisets and triangular fuzzy multiset", Mathematics **10** (2022) 726. https://doi.org/10.3390/math10050726.

[13] T. Sangodapo, “Some new operations on picture fuzzy multisets”, International Journal of Mathematical Sciences and Optimization: Theory and Applications **11** (2025) 19. https://doi.org/10.5281/zenodo.17579026.

[14] I. Couso & L. Godo, “Revisiting fuzzy multisets: clarifying ambiguities and the underlying algebraic structure”, Computational and Applied Mathematics **45** (2026) 156. https://doi.org/10.1007/s40314-025-03483-6.

[15] D. Ramot, R. Milo, M. Friedman & A. Kandel, “Complex fuzzy sets”, IEEE Transactions on Fuzzy Systems **10** (2002) 171. https://doi.org/10.1109/91.995119.

[16] G. Zhang, T. S. Dillon, K.-Y. Cai, J. Ma & J. Lu, “Operation properties and δ-equalities of complex fuzzy sets”, International Journal of Approximate Reasoning **50** (2009) 1227. https://doi.org/10.1016/j.ijar.2009.05.010.

[17] Y. Al-Qudah & N. Hassan, “Operations on complex multi-fuzzy sets”, Journal of Intelligent and Fuzzy Systems: Applications in Engineering and Technology **33** (2017) 1527. https://doi.org/10.3233/JIFS-162428.