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# Quasi-subordination for a subclass of non-Bazilevič functions connected with a $q$-derivative operator

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## Abstract

This study examines a broad subclass of non-Bazilevič functions that includes several subclasses of $q$-bounded turning functions and $q$-Sakaguchi functions. We link the definition of the class with a modified Opoola $q$-derivative operator, quasi-subordination, and a few number of mathematical concepts such as $q$-calculus and infinite series formations. Among the achievements in this work are the estimates for the early upper coefficient bounds and the Fekete-Szegö inequalities having complex parameters. In general, this unique class reduces to various recognized classes of non-Bazilevic functions when some of the parameters take values within their interval of definition.

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## 1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions that are analytic in the unit disk $\mathcal{E}$ and of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (z \in \mathcal{E} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}), \tag{1}$$

normalized such that $f(0) = f'(0) - 1 = 0$. More so, let '$\prec$' represent the well-known notation for *subordination*. Thus, for $f,\ F \in \mathcal{A}$, then $f \prec F$ if there exists an analytic function

$$\varepsilon(z) = \varepsilon_1 z + \varepsilon_2 z^2 + \varepsilon_3 z^3 + \cdots \quad (\varepsilon(0) = 0,\ |\varepsilon(z)| \leqslant |z| < 1,\ z \in \mathcal{E}), \tag{2}$$

such that $f(z) = F(\varepsilon(z))$. Suppose $F$ is univalent in $\mathcal{E}$, then

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{E}) \subset F(\mathcal{E}).$$

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In the sequel, $f$ is *quasi-subordinate* to $F$ (that is $f \preceq F$) if there exits an analytic function

$$\xi(z) = \xi_0 + \xi_1 z + \xi_2 z^2 + \dots \quad (|\xi(z)| \leqslant 1,\ z \in \mathcal{E}), \tag{3}$$

such that

$$f(z) = \xi(z) F(\varepsilon(z)). \tag{4}$$

Historically, Robertson [1] initiated the concept of quasi-subordination while clearly, when $\xi(z) = 1$ in Eq. (4), then quasi-subordination becomes subordination. More so, if $\varepsilon(z) = z$ in Eq. (4), then quasi-subordination becomes majorization (that is $f \ll F$), a concept initiated by MacGregor [2]. Obviously,

$$f(z) \preceq F(z) \implies f(z) = \xi(z)F(z) \implies f(z) \ll F(z) \quad (z \in \mathcal{E}).$$

For further information, see Ref. [3].

### 1.1. Non-Bazilevič functions

In the field of complex analysis, particularly within geometric function theory, Bazilevič functions, see Ref. [4], represent a class of analytic and univalent (one-to-one) functions defined in the open unit disk. These functions generalize many analytic and univalent functions such as starlike, convex, bounded turning, close-to-convex, and Yamaguchi functions which are characterized by differential inequalities involving some underlaying parameters and auxiliary functions.

However, non-Bazilevič functions are those analytic functions in the unit disk that do not satisfy the defining conditions of Bazilevič functions. Non-Bazilevič functions may still be analytic and univalent but do not adhere to the specific geometric and analytical properties required by the Bazilevič class. This distinction is significant because Bazilevič functions often exhibit strong geometric behavior such as mapping the unit disk onto convex, starlike, spirallike, etc. domains while non-Bazilevič functions may display more irregular or less constrained mappings.

The study of non-Bazilevič functions is important as it helps to understand the limitations of the Bazilevič class and broadens the scope of univalent function theory. These functions also provide counterexamples or test cases for conjectures in the geometric function theory. While Bazilevič functions have structured representation formulas and differential subordinations, non-Bazilevič functions lack these, making their analysis more complex and less predictable. Despite this, they remain a valuable subject of investigation in modern pure mathematical research. The class of non-Bazilevič functions was introduced by Obradović [5] and defined as functions that satisfy the conditions

$$\Re e \left\{ f'(z) \left( \frac{z}{f(z)} \right)^{1+\beta} \right\} > 0 \quad (0 < \beta < 1,\ z \in \mathcal{E}).$$

More details on subclasses of non-Bazilevič functions are accessible from Ref. [6].

### 1.2. Some classes of analytic functions

A class of functions that plays an essential role in the development of many classes of analytic (and univalent) functions is the class of Carathéodory functions. This class was introduced by Carathéodory in 1907 and defined as functions of the infinite series form

$$P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (\Re e\, P(z) > 0,\ P(0) = 1,\ z \in \mathcal{E}). \tag{5}$$

Let the class of Carathéodory functions be represented by $\mathcal{PT}$. An important function in class $\mathcal{PT}$ is the Möbius function

$$\mathcal{L}_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{j=1}^{\infty} z^j \quad (z \in \mathcal{E}), \tag{6}$$

or one of its rotation functions

$$\mathcal{L}_\delta(z) = \frac{1+e^{i\delta}z}{1-e^{i\delta}z} = 1 + 2\sum_{k=1}^{\infty} e^{ik\delta} z^k \quad (\delta \in \mathbb{R},\ z \in \mathcal{E}), \tag{7}$$

that maps the unit disk onto the whole *open-right-half-plane* of the complex plane. This property primly applauses it as the extremal function in $\mathcal{PT}$ where it plays a significant role in solving many extremal problems. In 2020, Umar *et al.* [7] introduced the subclass $\mathcal{PT}(c, A, B) \subseteq \mathcal{PT}$. A function $P(z)$ in Eq. (5) is said to be in the class $\mathcal{PT}(c, A, B)$ if it satisfies the subordination condition

$$1 + \frac{1}{c}[P(z) - 1] \prec \frac{1+Az}{1+Bz},$$

such that $c \in \mathbb{C} \setminus \{0\}$, $-1 \leqslant B < A \leqslant 1$, and $z \in \mathcal{E}$. Furthermore, the application of the subordination principle shows that

$$1 + \frac{1}{c}[P(z) - 1] = \frac{1 + A\varepsilon(z)}{1 + B\varepsilon(z)},$$

from where simple calculation gives

$$\begin{aligned} P(z) &= \frac{c(A-B)\varepsilon(z) + 1 + B\varepsilon(z)}{1 + B\varepsilon(z)} \\ &= 1 + c(A-B)z - cB(A-B)z^2 + \cdots \\ &= 1 + c(A-B)\sum_{j=1}^{\infty}(-B)^{j-1}z^j. \end{aligned} \tag{8}$$

We observe that $\mathcal{PT}(1,1,-1) = \mathcal{P}$, the class of functions in Eq. (5); $\mathcal{PT}(1,A,B) = \mathcal{P}(A,B)$, the class introduced and studied by Janowski [8]; and for $0 \leqslant \lambda < 1, \mathcal{PT}(1-\lambda,A,B) = \mathcal{P}(\lambda,A,B)$, the class introduced and studied by Polatoğlu *et al.* [9].

### 1.3. On certain q-differential operators

Jackson [10] introduced and studied many properties of the $q$-derivative operator $\mathfrak{D}_q$ defined as

$$\mathfrak{D}_q g(z) = \frac{g(z) - g(zq)}{(1-q)z} \quad (0 < q < 1).$$

Thus, for function $f \in \mathcal{A}$ of the form Eq. (1) and for $0 < q < 1$, the $q$-differential operator $\mathfrak{D}_q$ that maps $\mathcal{A}$ onto $\mathcal{A}$ is define on $f$ by

$$\left.\begin{aligned} &\mathfrak{D}_q f(0) = f'(0) = 1 \\ &\mathfrak{D}_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} = 1 + \sum_{j=2}^{\infty}[j]_q a_j z^{j-1} & (z \neq 0) \\ f'(z) \text{ as } q \longrightarrow 1 \end{cases} \\ &\mathfrak{D}_q^2 f(z) = \mathfrak{D}_q(\mathfrak{D}_q f(z)) = \sum_{j=2}^{\infty}[j-1]_q[j]_q a_j z^{j-2} \end{aligned}\right\}, \tag{9}$$

where

$$[j]_q = \frac{1-q^j}{1-q} = 1 + q + q^2 + q^3 + \cdots + q^{j-1}, \tag{10}$$

and clearly, $\lim_{q\longrightarrow 1}[j]_q = j$. The $q$-derivative operator have been widely applied by many researchers, see Refs. [11–16]. Using Eq. (9), we give a comprehensive definition of a modified Opoola $q$-derivative operator initiated by Alatawi and Darus [11].

**Definition 1.1.** Let $\mathcal{D}_{q,x,y}^{n,u,v}$ be a $q$-operator that maps $\mathcal{A}$ to $\mathcal{A}$ and defined as follows.

$$\begin{aligned} &\mathcal{D}_{q,x,y}^{0,u,v} f(z) = f(z), \\ &\mathcal{D}_{q,x,y}^{1,u,v} f(z) = yz\mathfrak{D}_q f(z) - yz(v-u) + z(1-x)(1-y) + [x + (v-u-x)y]f(z) = \varDelta_{q,y} f(z), \\ &\mathcal{D}_{q,x,y}^{2,u,v} f(z) = \varDelta_{q,y}(\mathcal{D}_{q,x,y}^{1,u,v} f(z)), \end{aligned}$$

so that in general,

$$\mathcal{D}_{q,x,y}^{n,u,v} f(z) = \varDelta_{q,y}(\mathcal{D}_{q,x,y}^{n-1,u,v} f(z)),$$

which corresponds to

$$\mathcal{D}_{q,x,y}^{n,u,v} f(z) = z + \sum_{j=2}^{\infty}[x + ([j]_q + v - u - x)y]^n a_j z^j \equiv z + \sum_{j=2}^{\infty}\varLambda_j a_j z^j, \tag{11}$$

where

$$\left.\begin{aligned} z \in \mathcal{E},\ n \in \{0,1,2,\ldots\},\ y \geqslant 0,\ v \geqslant 0,\ 0 \leqslant u + x \leqslant v, \\ q \in (0,1),\ [j]_q = \tfrac{1-q^j}{1-q},\ \text{and } \varLambda_j = [x + ([j]_q + v - u - x)y]^n \end{aligned}\right\}. \tag{12}$$

We however remark that the $q$-operator in Eq. (11) generalizes the following earlier known ($q$-)operators.

1. $\lim_{q\to 1} \mathcal{D}^{0,u,v}_{q,x,y} f(z) = \lim_{q\to 1} \mathcal{D}^{n,u,v}_{q,1,0} f(z) = f \in \mathcal{A}$ in Eq. (1).
2. $\lim_{q\to 1} \mathcal{D}^{n,u,u}_{q,x,1} f(z) = \lim_{q\to 1} \mathcal{D}^{n,v,v}_{q,x,1} f(z) = \mathcal{D}^{n} f(z)$ is the Sălăgean differential operator [17–21].
3. $\mathcal{D}^{n,u,u}_{q,x,1} f(z) = \mathcal{D}^{n,v,v}_{q,x,1} f(z) = \mathcal{D}^{n}_{q} f(z)$ is the Sălăgean $q$-differential operator [16].
4. $\lim_{q\to 1} \mathcal{D}^{n,u,u}_{q,x,y} f(z) = \lim_{q\to 1} \mathcal{D}^{n,v,v}_{q,x,y} f(z) = \mathcal{D}^{n}_{y} f(z)$ is the Al-Oboudi differential operator [22].
5. $\mathcal{D}^{n,u,u}_{q,x,y} f(z) = \mathcal{D}^{n,v,v}_{q,x,y} f(z) = \mathcal{D}^{n}_{q,y} f(z)$ is the Al-Oboudi $q$-differential operator [23].
6. $\lim_{q\to 1} \mathcal{D}^{n,u,v}_{q,1,y} f(z) = \mathcal{D}^{n,u,v}_{y} f(z)$ is the Opoola differential operator [22, 24–26].
7. $\mathcal{D}^{n,u,v}_{q,1,y} f(z)$ is the Opoola $q$-differential operator [13, 14].

## 2. Lemmas and proposition

**Lemma 2.1.** ([27, Lemma 2]). *Let $\varepsilon(z)$ be as defined in Eq. (2), then*

$$|\varepsilon_j| \leqslant 1 \quad (j \in \mathbb{N}).$$

**Lemma 2.2.** ([28, Lemma 2]). *Let $\varepsilon(z)$ be as defined in Eq. (2), then for $\Phi \in \mathbb{C}$,*

$$|\varepsilon_2 - \Phi\varepsilon_1^2| \leqslant \max\{1;\ |\Phi|\}.$$

**Proposition 2.3.** *Let $f \in \mathcal{A}$, then*

$$\frac{f(sz) - f(tz)}{s - t} = \frac{\left(sz + \sum_{j=2}^{\infty} a_j (sz)^j\right) - \left(tz + \sum_{j=2}^{\infty} a_j (tz)^j\right)}{s - t} = z + \sum_{j=2}^{\infty} \gamma_j a_j z^j,$$

*for*

$$\gamma_j = \frac{s^j - t^j}{s - t} \quad (j \in \{1, 2, 3, \ldots\}),$$

*so that*

$$\left(\frac{z(s-t)}{f(sz) - f(tz)}\right)^{\beta} = 1 - \beta\gamma_2 a_2 z + \left(\frac{(1+\beta)\beta}{2}\gamma_2^2 a_2^2 - \beta\gamma_3 a_3\right) z^2 + \cdots \tag{13}$$

*where we declare that*

$$z \in \mathcal{E},\ \beta \geqslant 0,\ \ s, t \in \mathbb{C} \setminus \{1\},\ \ s \neq t,\ \text{and}\ |t| \leqslant 1. \tag{14}$$

## 3. Core results

### 3.1. A novel class of analytic functions

Using the aforementioned details, we therefore introduce and study the properties of the following class of analytic functions.

**Definition 3.1.** A function $f \in \mathcal{A}$ of the form Eq. (1) is said to be a member of the class $\Upsilon^{n,u,v}_{q,x,y}(s, t, \alpha, \beta; c, A, B)$ if and only if it satisfies the geometric condition

$$(1-\alpha)\mathfrak{D}_q(\mathcal{D}^{n,u,v}_{q,x,y} f(z)) + \alpha\mathfrak{D}_q(\mathcal{D}^{n,u,v}_{q,x,y} f(z))\left(\frac{z(s-t)}{f(sz) - f(tz)}\right)^{\beta} - 1 \preceq [P(z) - 1], \tag{15}$$

where $0 \leqslant \alpha \leqslant 1$, and the declarations in Eq. (12) and Eq. (14) hold.

**Remark 3.2.** The following classes hold for some specific values of certain parameters.

1. If $q \to 1$ and $n = 0$ in Eq. (15), then we have the class of functions that satisfy the quasi-subordination condition

$$(1-\alpha) f'(z) + \alpha f'(z)\left(\frac{z(s-t)}{f(sz) - f(tz)}\right)^{\beta} - 1 \preceq [P(z) - 1].$$

   This class was studied by Shah *et al.* [6].

2. If $q \to 1$, $n = 0$ and $s = 1$ in Eq. (15), then we have the class of functions that satisfy the quasi-subordination condition

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{z(1-t)}{f(z)-f(tz)}\right)^{\beta} - 1 \precsim [P(z)-1].$$

3. If $q \to 1$, $n = 0$, and $s = 1$ in Eq. (15), then we have the class of functions that satisfy the quasi-subordination condition

$$(1-\alpha)f'(z) + \alpha f'(z)\left(\frac{z(1-t)}{f(z)-f(tz)}\right)^{\beta} - 1 \precsim \left[\frac{1+Az}{1+Bz} - 1\right].$$

   This class was studied Nunokwa *et al.* [29].

4. If $q \to 1$, $n = 0$, and $\alpha = 1$, in Eq. (15), then we have the class of functions that satisfy the quasi-subordination condition

$$f'(z)\left(\frac{z(s-t)}{f(sz)-f(tz)}\right)^{\beta} - 1 \precsim [P(z)-1].$$

5. If $q \to 1$, $n = 0 = \alpha$ (or $\alpha - 1 = 0 = \beta$) in Eq. (15), then we have the class of functions that satisfy the quasi-subordination condition

$$f'(z) - 1 \precsim [P(z)-1].$$

6. If $q \to 1$, $n = 0$, and $s = \alpha = 1$ in Eq. (15), then we have the class of functions that satisfy the quasi-subordination condition

$$f'(z)\left(\frac{z(1-t)}{f(z)-f(tz)}\right)^{\beta} - 1 \precsim [P(z)-1].$$

   This class was studied by Sharma and Raina [30].

Closely related to the quasi-subordination condition Eq. (15) are:

7.
$$\Re e\left(\frac{z(1-t)f'(z)}{f(z)-f(tz)}\right) > \sigma,$$

   where $z \in \mathcal{E}$, $0 \leqslant \sigma < 1$, $t \in \mathbb{C} \setminus \{1\}$, $|t| \leqslant 1$, and studied by Owa *et al.* [31].

8.
$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{z(1-t)}{f(z)-f(tz)}\right)^{\beta} - 1 \precsim [P(z)-1],$$

   where $0 \leqslant \alpha \leqslant 1, \beta \geqslant 0$, $t \in \mathbb{C} \setminus \{1\}$, $|t| \leqslant 1$, and studied by Srivastava *et al.* [32].

9.
$$\Re e\left(\frac{zf(z)}{f(z)-f(z)}\right) > 0,$$

   where $z \in \mathcal{E}$ and studied by Sakaguchi [33].

In this work, we introduce a new subclass of non-Bazilevic functions defined by modified Opoola $q$-derivative operator and quasi-subordination. We thereafter explore some coefficient estimate properties and the Fekete-Szegö functional with complex parameters for the class. Henceforth to avoid repetition, let all parameter be as defined in Eq. (12) and Eq. (14).

*3.2. Coefficient bound estimates*

**Theorem 3.3.** *If $f \in \mathcal{A}$ belongs to the class $\Upsilon_{q,x,y}^{n,u,v}(s,t,\alpha,\beta;c,A,B)$, then*

$$|a_2| \leqslant \frac{|c|(A-B)|\xi_0|}{|[2]_q\Lambda_2 - \alpha\beta\gamma_2|}, \tag{16}$$

*and*

$$|a_3| \leqslant \frac{|c|(A-B)}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|}\Big[|\xi_0| \max\{1; |\Phi|\} + |\xi_1|\Big], \tag{17}$$

*where*

$$\Phi = B + \frac{\alpha\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2}. \tag{18}$$

*Proof.* Let $f \in \mathcal{A}$, then the application of quasi-subordination principle in Eq. (15) shows that

$$(1-\alpha)\mathfrak{D}_q(\mathcal{D}^{n,u,v}_{q,x,y}f(z)) + \alpha\mathfrak{D}_q(\mathcal{D}^{n,u,v}_{q,x,y}f(z))\left(\frac{(s-t)z}{f(sz)-f(tz)}\right)^{\beta} - 1 = \xi(z)[P(\varepsilon(z))-1]. \tag{19}$$

Putting Eq. (2), Eq. (3), and Eq. (8) into Eq. (19) gives the expansion

$$RHS = c(A-B)\varepsilon_1\xi_0 z + \{c(A-B)(\varepsilon_2 - B\varepsilon_1^2)\xi_0 + c(A-B)\varepsilon_1\xi_1\}z^2 + \cdots \tag{20}$$

and, using Eq. (9), Eq. (11), and Eq. (13) in Eq. (19) gives the expansion

$$LHS = ([2]_q\Lambda_2 - \alpha\beta\gamma_2)a_2 z + \left\{\left(\frac{\alpha\beta(1+\beta)\gamma_2^2}{2} - [2]_q\alpha\beta\gamma_2\Lambda_2\right)a_2^2 + ([3]_q\Lambda_3 - \alpha\beta\gamma_3)a_3\right\}z^2 + \cdots. \tag{21}$$

Further, the comparison of the terms in Eq. (20) and Eq. (21) shows that

$$([2]_q\Lambda_2 - \alpha\beta\gamma_2)a_2 = c(A-B)\varepsilon_1\xi_0,$$

where simple rearrangement yields

$$a_2 = \frac{c(A-B)\varepsilon_1\xi_0}{[2]_q\Lambda_2 - \alpha\beta\gamma_2}, \tag{22}$$

to give

$$|a_2| \leqslant \frac{|c|(A-B)|\varepsilon_1||\xi_0|}{|[2]_q\Lambda_2 - \alpha\beta\gamma_2|},$$

so that the application of Lemma 2.1 gives the result in Eq. (16). Again, from Eq. (20) and Eq. (21) we get

$$\left(\frac{\alpha\beta(1+\beta)\gamma_2^2}{2} - [2]_q\alpha\beta\gamma_2\Lambda_2\right)a_2^2 + ([3]_q\Lambda_3 - \alpha\beta\gamma_3)a_3 = c(A-B)(\varepsilon_2 - B\varepsilon_1^2)\xi_0 + c(A-B)\varepsilon_1\xi_1,$$

where the substitution for $a_2^2$ and further simplification lead to

$$a_3 = \frac{c(A-B)\{(\varepsilon_2 - B\varepsilon_1^2)\xi_0 + \varepsilon_1\xi_1\}}{([3]_q\Lambda_3 - \alpha\beta\gamma_3)} - \frac{\alpha\beta\gamma_2 c^2(A-B)^2\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\varepsilon_1^2\xi_0^2}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2([3]_q\Lambda_3 - \alpha\beta\gamma_3)},$$

to get

$$a_3 = \frac{c(A-B)\varepsilon_2\xi_0}{[3]_q\Lambda_3 - \alpha\beta\gamma_3} - \frac{cB(A-B)\varepsilon_1^2\xi_0}{[3]_q\Lambda_3 - \alpha\beta\gamma_3} + \frac{c(A-B)\varepsilon_1\xi_1}{[3]_q\Lambda_3 - \alpha\beta\gamma_3} - \frac{\alpha\beta\gamma_2 c^2(A-B)^2\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\varepsilon_1^2\xi_0^2}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2([3]_q\Lambda_3 - \alpha\beta\gamma_3)}, \tag{23}$$

which further simplifies to

$$a_3 = \frac{c(A-B)\xi_0}{[3]_q\Lambda_3 - \alpha\beta\gamma_3}\left\{\varepsilon_2 - \left[B + \frac{\alpha\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2}\right]\varepsilon_1^2\right\} + \frac{c(A-B)\varepsilon_1\xi_1}{[3]_q\Lambda_3 - \alpha\beta\gamma_3}.$$

Now,

$$|a_3| \leqslant \frac{|c|(A-B)|\xi_0|}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|}\left|\varepsilon_2 - \left[B + \frac{\alpha\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2}\right]\varepsilon_1^2\right| + \frac{|c|(A-B)|\varepsilon_1||\xi_1|}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|},$$

or

$$|a_3| \leqslant \frac{|c|(A-B)|\xi_0|}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|}\left|\varepsilon_2 - \Phi\varepsilon_1^2\right| + \frac{|c|(A-B)|\varepsilon_1||\xi_1|}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|},$$

where $\Phi$ is as declared in Eq. (18). Further, the application of Lemmas 2.1 and 2.2, and some simplifications give the result in Eq. (17). □

We have the following corollaries for some specific values of $\alpha$.

**Corollary 3.4.** *If $f \in \mathcal{A}$ belongs to the class $\Upsilon^{n,u,v}_{q,x,y}(s,t,\alpha,\beta;c,A,B)$, then for $\alpha = 0$,*

$$|a_2| \leqslant \frac{|c|(A-B)|\xi_0|}{[2]_q\Lambda_2},$$

*and*

$$|a_3| \leqslant \frac{|c|(A-B)}{[3]_q\Lambda_3}\Big[|\xi_0|\max\{1;|B|\} + |\xi_1|\Big].$$

**Corollary 3.5.** *If $f \in \mathcal{A}$ belongs to the class $\Upsilon_{q,x,y}^{n,u,v}(s,t,\alpha,\beta;c,A,B)$, then for $\alpha = 1$,*

$$|a_2| \leqslant \frac{|c|(A-B)|\xi_0|}{|[2]_q\Lambda_2 - \beta\gamma_2|},$$

*and*

$$|a_3| \leqslant \frac{|c|(A-B)}{|[3]_q\Lambda_3 - \beta\gamma_3|}\Big[|\xi_0| \max\{1; |\Omega|\} + |\xi_1|\Big],$$

*where*

$$\Omega = B + \frac{\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \beta\gamma_2)^2}.$$

### *3.3. Fekete-Szegö coefficient estimates*

In 1933, Fekete-Szeg [34] established a precise upper bound for the functional

$$\varphi(\tau, f) = |a_3 - \tau a_2^2|, \tag{24}$$

where $\tau \in \mathbb{R}$ and $a_2$ and $a_3$ are $f$ coefficients in (1). Many academics have studied this functional for various subclasses of $\mathcal{A}$, as evidenced by the volume of literature since its introduction. In 1986, Pfluger [35] presented the functional (24) with complex parameter $\tau$. Literally, the Bieberbach conjecture serves as the foundation for the analysis of this function. See Ref. [36] for further information.

**Theorem 3.6.** *If $f \in \mathcal{A}$ belongs to the class $\Upsilon_{q,x,y}^{n,u,v}(s,t,\alpha,\beta;c,A,B)$, then*

$$\varphi(\tau, f) \leqslant \frac{|c|(A-B)}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|}\Big[|\xi_0| \max\{1; |\Psi|\} + |\xi_1|\Big],$$

*for $\tau \in \mathbb{C}$, and*

$$\Psi = B + \frac{\alpha\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2} + \frac{\tau c(A-B)([3]_q\Lambda_3 - \alpha\beta\gamma_3)\xi_0}{([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2}. \tag{25}$$

*Proof.* Using Eq. (22) and Eq. (23) in Eq. (24) means

$$a_3 - \tau a_2^2 = \frac{c(A-B)\varepsilon_2\xi_0}{[3]_q\Lambda_3 - \alpha\beta\gamma_3} - \frac{cB(A-B)\varepsilon_1^2\xi_0}{[3]_q\Lambda_3 - \alpha\beta\gamma_3} + \frac{c(A-B)\varepsilon_1\xi_1}{[3]_q\Lambda_3 - \alpha\beta\gamma_3} - \frac{\alpha\beta\gamma_2 c^2(A-B)^2\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\varepsilon_1^2\xi_0^2}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2([3]_q\Lambda_3 - \alpha\beta\gamma_3)} - \tau\left(\frac{c(A-B)\varepsilon_1\xi_0}{[2]_q\Lambda_2 - \alpha\beta\gamma_2}\right)^2,$$

where we further simplify it to achieve the equation

$$a_3 - \tau a_2^2 = \frac{c(A-B)\xi_0}{[3]_q\Lambda_3 - \alpha\beta\gamma_3}\left\{\varepsilon_2 - \left[B + \frac{\alpha\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2} + \frac{\tau c(A-B)([3]_q\Lambda_3 - \alpha\beta\gamma_3)\xi_0}{([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2}\right]\varepsilon_1^2\right\} + \frac{c(A-B)\varepsilon_1\xi_1}{[3]_q\Lambda_3 - \alpha\beta\gamma_3}.$$

Now,

$$\left|a_3 - \tau a_2^2\right| \leqslant \frac{|c|(A-B)|\xi_0|}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|}\left|\varepsilon_2 - \Psi\varepsilon_1^2\right| + \frac{|c|(A-B)|\varepsilon_1||\xi_1|}{|[3]_q\Lambda_3 - \alpha\beta\gamma_3|},$$

where $\Psi$ is as declared in Eq. (25). Thus, the application of Lemmas 2.1 and 2.2 gives the result in the theorem. □

We have the following corollaries for some specific values of $\alpha$.

**Corollary 3.7.** *If $f \in \mathcal{A}$ belongs to the class $\Upsilon_{q,x,y}^{n,u,v}(s,t,\alpha,\beta;c,A,B)$, then for $\alpha = 0$,*

$$\varphi(\tau, f) \leqslant \frac{|c|(A-B)}{[3]_q\Lambda_3}\Big[|\xi_0| \max\{1; |\Theta|\} + |\xi_1|\Big],$$

*for $\tau \in \mathbb{C}$, and*

$$\Theta = B + \frac{\tau c[3]_q\Lambda_3(A-B)\xi_0}{[2]_q^2\Lambda_2^2}.$$

**Corollary 3.8.** *If $f \in \mathcal{A}$ belongs to the class $\Upsilon^{n,u,v}_{q,x,y}(s,t,\alpha,\beta;c,A,B)$, then for $\alpha = 1$,*

$$\varphi(\tau, f) \leqslant \frac{|c|(A-B)}{|[3]_q\Lambda_3 - \beta\gamma_3|}\Big[|\xi_0| \max\{1; |\vartheta|\} + |\xi_1|\Big],$$

*for $\tau \in \mathbb{C}$, and*

$$\vartheta = B + \frac{\beta\gamma_2 c(A-B)\{(1+\beta)\gamma_2 - 2[2]_q\Lambda_2\}\xi_0}{2([2]_q\Lambda_2 - \alpha\beta\gamma_2)^2} + \frac{\tau c(A-B)([3]_q\Lambda_3 - \beta\gamma_3)\xi_0}{([2]_q\Lambda_2 - \beta\gamma_2)^2}.$$

## 4. Conclusion

In this work, we explored a subclass of non-Bazilevič functions that included several subclasses. The definition of the new class was a combination of a modified Opoola $q$-derivative operator, quasi-subordination, and $q$-calculus. Among the results obtained are the upper bound of the coefficient estimate and the Fekete-Szegö inequalities with complex parameters for the class. The new class reduces to various known subclasses of non-Bazilevič functions when some of the the parameters take values within their interval of definition.

## Data availability

This research did not generate or analyze any datasets. As such, data sharing is not applicable.

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## References

[1] M. S. Robertson, "Quasi-subordination and coefficients conjectures", Bulletin of American Mathematical Society **76** (1970) 1. https://doi.org/10.1090/S0002-9904-1970-12356-4.
[2] T. H. MacGregor, "Majorization by univalent functions", Duke Mathematical Journal **34** (1967) 1. https://doi.10.1215/S0012-7094-67-03411-4.
[3] R. O. Ayinla & A. O. Lasode, "Estimates for a class of analytic and univalent functions connected with quasi-subordination", Proceedings of the Nigerian Society of Physical Sciences Conferences **1** (2024) 82. https://doi.org/10.61298/pnspsc.2024.1.82.
[4] I. E. Bazilevič, "On a case of integrability by quadratures of the equation of Loewner Kufarev", Matematicheskii Sbornik **37** (1955) 79. http://mi.mathnet.ru/eng/sm5227.
[5] M. Obradovic, "A class of univalent functions", Hokkaido Mathematical Journal **27** (1998) 2. https://doi.org/10.14492/HOKMJ/1351001289.
[6] S. G. Shah, S. Al-Sa'di, S. Hussain, A. Tasleem, A. Rasheed, I. Z. Cheema & M. Darus, "Fekete-Szegö functional for a class of non-Bazilevič functions related to quasi-subordination", Demonstratio Mathematica **56** (2023) 1. https://doi.org/10.1515/dema-2022-0232.
[7] S. Umar, M. Arif, M. Raza & S. K. Lee, "On a subclass related to Bazilevič functions", AIMS Mathematics **5** (2020) 3. https://doi.org/10.3934/math.2020135.
[8] W. Janowski, "Some extremal problems for certain families of analytic functions I", Annales Polonici Mathematici **28** (1973) 3. https://doi.org/10.4064/ap-28-3-297-326.
[9] Y. Polatoğlu, S. M. Bolcal, A. Şen & E. Yavuz, "A study on the generalization of Janowski functions in the unit disc", Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **22** (2006) 27. https://www.emis.de/journals/AMAPN/vol22_1/amapn22_04.pdf.
[10] F. H. Jackson, "On $q$-functions and a certain difference operator", Transactions of the Royal Society of Edinburgh **46** (1908) 2. https://doi.org/10.1017/S0080456800002751.
[11] A. A. Alatawi & M. Darus, "On a certain subclass of analytic functions involving the modified $q$-Opoola derivative operator", International Journal of Nonlinear Analysis and Applications **14** (2023) 5. http://dx.doi.org/10.22075/ijnaa.2023.29137.4072.
[12] A. Aral, V. Gupta & R. P. Agarwal, *Applications of q-calculus in operator theory,* Springer Science+Business Media, New York, 2013, pp. 3–5. https://doi.org/10.1007/978-1-4614-6946-9_1.
[13] A. O. Lasode & T. O. Opoola, "Some properties of a family of univalent functions defined by a generalized Opoola differential operator", General Mathematics **30** (2022) 1. https://doi.org/10.2478/gm-2022-0001.
[14] A. O. Lasode & T. O. Opoola, "Some properties of a class of generalized Janowski-type $q$-starlike functions associated with Opoola $q$-differential operator and $q$-differential subordination", Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics **73** (2024) 2. https://doi.org/10.31801/cfsuasmas.1281348.
[15] A. Saliu & S. O. Oladejo, "On lemniscate of Bernoulli of $q$-Janowski type", Journal of the Nigerian Society of Physical Sciences **4** (2022) 961. https://doi.org/10.46481/jnsps.2022.961.
[16] M. Govindaraj & S. Sivasubramanian, "On a class of analytic functions related to conic domains involving $q$-calculus", Analysis Mathematica **43** (2017) 3. https://doi.org/10.1007/s10476-017-0206-5.
[17] R. A. Bello, "Coefficient estimate for a subclass of analytic functions defined by a generalized differential operator", International Journal of Advances in Scientific Research and Engineering **9** (2023) 7. https://doi.org/10.31695/IJASRE.2023.9.7.8.
[18] R. A. Bello & T. O. Opoola, "On some properties of a class of analytic Functions defined by Sălăgean differential operator", Journal of Progressive Research in Mathematics **18** (2021) 3. https://scitecresearch.com/ojs-3.2.1-5/index.php/jprm/article/view/1985.
[19] L. M. Fatunsin, "On a new subclass of univalent functions defined by Sălăgean differential operator", Numerical Analysis and Applicable Mathematics **3** (2022) 25. https://doi.org/10.36686/Ariviyal.NAAM.2022.03.06.014.
[20] G. S. Sălăgean, "Subclasses of univalent functions", Lecture Notes in Mathematics **1013** (1983) 362. https://doi.org/10.1007/BFb0066543.

[21] R. O. Ayinla & R. A. Bello, "Toeplitz determinants for a subclass of analytic functions.", Journal of Progressive Research Mathematics **18** (2021) 1. http://scitecresearch.com/journals/index.php/jprm/article/view/2034/1426.

[22] F. M. Al-Oboudi, "On univalent functions defined by a generalised Sălăgean operator", International Journal of Mathematics and Mathematical Sciences **2004** (2004) 27. http://dx.doi.org/10.1155/S0161171204108090.

[23] M. K. Aouf & F. Z. El-Emam, "Fekete-Szegö problems for certain classes of meromorphic functions involving $q$-Al-Oboudi differential operator", Journal of Mathematics **2022** (2022) 1. https://doi.org/10.1155/2022/4731417.

[24] T. O. Opoola, "On a subclass of univalent functions defined by a generalised differential operator", International Journal of Mathematical Analysis **11** (2017) 18. https://doi.org/10.12988/ijma.2017.7232.

[25] L. M. Fatunsin & T. O. Opoola, "New results on subclasses of analytic functions defined by Opoola differential operator", Journal of Mathematics and System Science **7** (2017) 295. https://doi.org/10.17265/2159-5291/2017.10.003.

[26] R. A. Bello, "On some properties of a class of analytic functions defined by Opoola differential operator", International Journal of Mathematics and Statistics Studies **12** (2023) 4. https://doi.org/10.37745/ijmss.13/vol12n45261.

[27] H. Shiraishi & T. Hayami, "Coefficient estimates for Schwarz functions", (2013) 1. https://doi.org/10.48550/arXiv.1302.6916.

[28] R. M. Ali, V. Ravichandran & N. Seenivasagan, "Coefficient bounds for $p$-valent functions", Applied Mathematics and Computation **187** (2007) 11. https://doi.org/10.1016/j.amc.2006.08.100.

[29] M. Nunokawa, M. Obradovič & S. Owa, "One criterion for univalency", Proceedings of the American Mathematical Society **106** (1989) 4. https://doi.org/10.1090/S0002-9939-1989-0975653-5.

[30] P. Sharma & R. K. Raina, "On a Sakaguchi-type class of univalent functions associated with quasi-subordination", Commentarii Mathematici Universitatis Sancti Pauli **64** (2015) 59. https://core.ac.uk/download/pdf/293152375.pdf.

[31] S. Owa, T. Sekine & R. Yamakawa, "On Sakaguchi-type functions", Applied Mathematics and Computation **187** (2007) 133. https://doi.org/10.1016/j.amc.2006.08.133.

[32] H. M. Srivastava, S. Hussain, A. Raziq & M. Raza, "The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination", Carpathian Journal of Mathematics **34** (2018) 1. http://www.jstor.org/stable/90021608.

[33] K. Sakaguchi, "On a certain univalent mapping", Journal of the Mathematical Society of Japan **11** (1959) 1. https://doi.org/10.2969/jmsj/01110072.

[34] M. Fekete & G. Szegö, "Eine bemerkung über ungerade schlichte funktionen", Journal of the London Mathematical Society **8** (1933) 85. https://doi.org/10.1112/jlms/s1-8.2.85.

[35] A. Pfluger, "The Fekete-Szegö inequality for complex parameters", Complex Variables, Thoery and Application: An International Journal **7** (1986) 149. https://doi.org/10.1080/17476938608814195.

[36] A. O. Lasode, R. O. Ayinla, R. A. Bello, A. A. Amao, L. M. Fatunsin, B. Sambo & O. Awoyale, "Hankel determinants with Fekete-Szegö parameter for a subset of Bazilevič functions linked with Ma-Minda function", Open Journal of Mathematical Analysis **9** (2025) 1. https://doi.org/10.30538/psrp-oma2025.0150.