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Approximate solution of higher-order oscillatory differential equations via modified linear block techniques

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Abstract

In many areas of science and engineering, problems modeled by ordinary differential equations (ODEs) often lack analytical solutions, requiring the use of numerical methods for approximation. The proposed method tackles the challenges of solving higher-order oscillatory differential equations and introduces a new linear block technique for directly solving these equations. This method improves accuracy, reduces computational effort, and simplifies coding complexity compared to previous approaches. A generalized algorithm is presented to derive the proposed method, which enhances existing techniques for second-order and higher-order oscillatory problems. The basic properties of the method were numerically analyzed, confirming its accuracy, stability, consistency, and convergence. The proposed method proves to be efficient and suitable for various test problems, including real-life problems, which demonstrate its accuracy as compared to the existing methods.

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1. Introduction

In science and engineering, many problems modeled with ordinary differential equations do not have exact analytical solutions, making it essential to rely on approximate solutions. These approximations are obtained using numerical methods, which lead to the discretization of the solutions. Discretization represents the solution as values of the function at specific grid points, and these values are connected by interpolating the function, as highlighted in studies such as Refs. [1, 2]. The resulting oscillatory differential equations (OSDE) give rise to higher-order differential equations (HDE) expressed in the form:

$$y^{(n)} = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right), \quad y^{(m-1)}(t_0) = \mu_{m-1}, \quad m = 1, 2, \ldots, (n-1), \tag{1}$$

are considered in this study, where $\mu_{m-1},\ m = 1, 2, \ldots, (n-1)$ are constants. Many physical issues remain underexplored and inadequately addressed by researchers. While some challenges in fields such as social sciences, technology, and science have

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received attention, numerous others are still awaiting thorough investigation. Oscillatory phenomena play a crucial role across these areas, and differential equations are fundamental for modeling such behaviours, as discussed in Refs. [3, 4].

Historically, solving Eq. (1) involved reducing it to a system of first-order ODEs and applying numerical methods designed for first-order systems, as outlined in Ref. [5]. However, this approach posed significant challenges, including high computational demands, increased manual effort, coding complexity, and reduced accuracy due to error accumulation. Researchers, such as those in Refs. [6–8], have extensively discussed these limitations. In response, direct methods for solving higher-order differential equations were proposed in Refs. [7–12]. As a result, various efforts have been made to develop numerical schemes that solve Eq. (1) directly using different approaches. Notably, Refs. [13–15] have proposed methods specifically designed to address second-order oscillatory problems of this form:

$$y'' = f(t, y, y'), \quad y(t_0) = \mu_0, \quad y'(t_1) = \mu_1. \tag{2}$$

Likewise, methods have been introduced by Refs. [16–18] to address the solution of

$$y''' = f(t, y, y', y''), \quad y(t_0) = \mu_0, \quad y'(t_1) = \mu_1, \quad y''(t_2) = \mu_2. \tag{3}$$

Finally, certain researchers Refs. [5, 9, 12, 19] developed different methods to solve:

$$y'''' = f(t, y, y', y'', y'''), \quad y(t_0) = \mu_0, \quad y'(t_1) = \mu_1, \quad y''(t_2) = \mu_2, \quad y'''(t_3) = \mu_3. \tag{4}$$

Similarly, other methods used by different researchers include the following: Ref. [20] employed a one-eighth step hybrid block method (OSHBM) to solve second-order initial value problems of ODEs. In Ref. [21], a third derivative hybrid block method (TDHBM) was used to obtain approximate solutions for third-order initial value problems. Ref. [22] simulated third-order linear problems using a single-step block method (SSBM), while Ref. [23] employed a monohybrid point linear multistep method (MH-PLMM) to solve nth-order ODEs. Finally, Ref. [24] applied a two-step hybrid block method (THBM) to handle higher-order initial value problems of ODEs.

The proposed method aim to improve the accuracy and effectiveness of the existing methods discussed in Refs. [4, 13–19, 25, 26] while enhancing the partitioning for greater efficiency.

The article is organized as follows: In Section 2, we develop a generalized algorithm for the n-step linear block techniques. In Section 3, we formulate the scheme for the proposed method and prove some corollaries. Section 4 presents numerical problems, along with computational results and graphical representations, to demonstrate the accuracy of the proposed method. The final section, Section 5, provides a discussion of the results, a summary, and concluding remarks.

2. Construction of linear block technique

The proposed linear block technique was derived based on the methods outlined in Refs. [3, 5, 12]. According to Ref. [12], the linear block technique was first proposed by Adeyeye and Omar [27], for solving second order initial value problems. This study adopts the linear block approach using $\left(0\left(\frac{1}{4}\right)\ 2\right)$ partition for the direct solution of Eq. (1).

2.1. Generalization algorithm for the k-step

The linear block technique was utilized in deriving a new method for the direct solution of Eq. (1), where $Y_{n+k} = (y_{n+a},\ y_{n+b},\ \ldots,\ y_{n+k})$ and $\left(y^{(j)}_{n+a},\ y^{(j)}_{n+b},\ \ldots,\ y^{(j)}_{n+k}\right)$. The unknowns are obtained by considering the generalized algorithm:

$$y_{n+\xi} = \sum_{j=0}^{3} \frac{(\xi h)^j}{j!} y_n^{(j)} + \sum_{j=0}^{k} \left(\psi_{i\xi} f_{n+j}\right), \quad \zeta = a,\ b,\ \ldots,\ k, \tag{5}$$

its higher derivatives

$$y^{d}_{n+\xi} = \sum_{j=0}^{4-(\varsigma+1)} \frac{(\xi h)^j}{j!} y_n^{(j+\varsigma)} + \varsigma = 1_{(\xi=a,\ b,\ \ldots,\ k)},\ 2_{(\xi=a,\ b,\ \ldots,\ k)},\ 3_{(\xi=a,\ b,\ \ldots,\ k)}, \tag{6}$$

with $\psi_{\xi j} = U^{-1}G$ and $\Omega_{\xi j} = U^{-1}D$ where

$$U = \begin{pmatrix} 1 & 1 & 1 & \cdots & k \\ 0 & \frac{(ah)^1}{1!} & \frac{(bh)^1}{1!} & \cdots & \frac{(kh)^1}{1!} \\ 0 & \frac{(ah)^2}{2!} & \frac{(bh)^2}{2!} & \cdots & \frac{(kh)^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{(ah)^m}{m!} & \frac{(bh)^m}{m!} & \cdots & \frac{(kh)^m}{m!} \end{pmatrix}, \quad G = \begin{pmatrix} \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \vdots \\ \frac{(\xi h)^{4+m}}{(4+m)!} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{(\xi h)^{4-\varsigma}}{(4-\varsigma)!} \\ \frac{(\xi h)^{(5-\varsigma)+a}}{((5-\varsigma)+a)!} \\ \frac{(\xi h)^{(6-\varsigma)+b}}{((6-\varsigma)+b)!} \\ \vdots \\ \frac{(\xi h)^{(m-\varsigma)+k}}{((m-\varsigma)+k)!} \end{pmatrix}.$$

So, to derive the new methods, the subsequent Corollary were proved.

Corollary 1

The k-multistep method associated with Eq. (5) uses only a block method. The corollary is extended to develop a higher-order scheme based on the block algorithm. This can be confirmed using Eqs. (5) and (6) as a block at the specified points: $\left(0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1,\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4},\ 2\right)$.

Proof

Now simplifying (5) and (6) using the partitioned points, we have:

$$U=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & \frac{h}{8} & \frac{h}{4} & \frac{3h}{4} & \frac{h}{2} & \frac{5h}{4} & \frac{3h}{2} & \frac{7h}{4} & 2h\\ 0 & \frac{(h/4)^2}{2!} & \frac{(h/2)^2}{2!} & \frac{(3h/4)^2}{2!} & \frac{h^2}{2!} & \frac{(5h/4)^2}{2!} & \frac{(3h/2)^2}{2!} & \frac{(7h/4)^2}{2!} & \frac{(2h)^2}{2!}\\ 0 & \frac{(h/4)^3}{3!} & \frac{(h/2)^3}{3!} & \frac{(3h/4)^3}{3!} & \frac{h^3}{3!} & \frac{(5h/4)^3}{3!} & \frac{(3h/2)^3}{3!} & \frac{(7h/4)^3}{3!} & \frac{(2h)^3}{3!}\\ 0 & \frac{(h/4)^4}{4!} & \frac{(h/2)^4}{4!} & \frac{(3h/4)^4}{4!} & \frac{h^4}{4!} & \frac{(5h/4)^4}{4!} & \frac{(3h/2)^4}{4!} & \frac{(7h/4)^4}{4!} & \frac{(2h)^4}{4!}\\ 0 & \frac{(h/4)^5}{5!} & \frac{(h/2)^5}{5!} & \frac{(3h/4)^5}{5!} & \frac{h^5}{5!} & \frac{(5h/4)^5}{5!} & \frac{(3h/2)^5}{5!} & \frac{(7h/4)^5}{5!} & \frac{(2h)^5}{5!}\\ 0 & \frac{(h/4)^6}{6!} & \frac{(h/2)^6}{6!} & \frac{(3h/4)^6}{6!} & \frac{h^6}{6!} & \frac{(5h/4)^6}{6!} & \frac{(3h/2)^6}{6!} & \frac{(7h/4)^6}{6!} & \frac{(2h)^6}{6!}\\ 0 & \frac{(h/4)^7}{7!} & \frac{(h/2)^7}{7!} & \frac{(3h/4)^7}{7!} & \frac{h^7}{7!} & \frac{(5h/4)^7}{7!} & \frac{(3h/2)^7}{7!} & \frac{(7h/4)^7}{7!} & \frac{(2h)^7}{7!}\\ 0 & \frac{(h/4)^8}{8!} & \frac{(h/2)^8}{8!} & \frac{(3h/4)^8}{8!} & \frac{h^8}{8!} & \frac{(5h/4)^8}{8!} & \frac{(3h/2)^8}{8!} & \frac{(7h/4)^8}{8!} & \frac{(2h)^8}{8!}\end{bmatrix},\ G=\begin{bmatrix}\frac{(\xi h)^4}{4!}\\ \frac{(\xi h)^5}{5!}\\ \frac{(\xi h)^6}{6!}\\ \frac{(\xi h)^7}{7!}\\ \frac{(\xi h)^8}{8!}\\ \frac{(\xi h)^9}{9!}\\ \frac{(\xi h)^{10}}{10!}\\ \frac{(\xi h)^{11}}{11!}\\ \frac{(\xi h)^{12}}{12!}\end{bmatrix},\ D=\begin{bmatrix}\frac{(\xi h)^{4-\tau}}{(4-\tau)!}\\ \frac{(\xi h)^{5-\tau}}{(5-\tau)!}\\ \frac{(\xi h)^{6-\tau}}{(6-\tau)!}\\ \frac{(\xi h)^{7-\tau}}{(7-\tau)!}\\ \frac{(\xi h)^{8-\tau}}{(8-\tau)!}\\ \frac{(\xi h)^{9-\tau}}{(9-\tau)!}\\ \frac{(\xi h)^{10-\tau}}{(10-\tau)!}\\ \frac{(\xi h)^{11-\tau}}{(11-\tau)!}\\ \frac{(\xi h)^{12-\tau}}{(12-\tau)!}\end{bmatrix}.$$

Solving Eqs. (5) and (6), we have:

$$y_n^{\xi},\quad \xi=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2.$$

Substituting $\xi=\xi_n+xh$, the polynomial takes the form:

$$\begin{aligned}y(\xi_n+xh)=&\ \alpha_{1/4}y_{n+1/4}+\alpha_1 y_{n+1}+\alpha_{3/2}y_{n+3/2}+\alpha_2 y_{n+2}\\ &+h^4(\beta_0 f_n+\beta_{1/4}f_{n+1/4}+\beta_{1/2}f_{n+1/2}+\beta_{3/4}f_{n+3/4}+\beta_1 f_{n+1}\\ &+\beta_{5/4}f_{n+5/4}+\beta_{3/2}f_{n+3/2}+\beta_{7/4}f_{n+7/4}+\beta_2 f_{n+2}),\end{aligned}\tag{7}$$

where

$$\begin{aligned}\alpha_{1/4}&=\frac{64}{35}-\frac{416}{35}\xi+\frac{96}{35}\xi^2-\frac{64}{105}\xi^3,\\ \alpha_1&=-2+\frac{31}{3}\xi-10\xi^2+\frac{8}{3}\xi^3,\\ \alpha_{3/2}&=\frac{8}{5}-\frac{44}{5}\xi+\frac{52}{5}\xi^2-\frac{16}{5}\xi^3,\\ \alpha_2&=-\frac{3}{7}+\frac{17}{7}\xi-\frac{22}{7}\xi^2+\frac{8}{7}\xi^3.\end{aligned}$$

$$\begin{aligned}
\beta_0 =\ & \frac{1}{23224320} + \frac{5200825781}{7664025600}\xi + \frac{14625071}{7664025600}\xi^2 - \frac{4165541}{348364800}\xi^3 + \frac{1}{24}\xi^4 - \frac{761}{8400}\xi^5 + \frac{29531}{226800}\xi^6 \\
& - \frac{89}{700}\xi^7 - \frac{4}{105}\xi^9 + \frac{52}{4725}\xi^{10} - \frac{32}{17325}\xi^{11} + \frac{62}{467775}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_{1/4} =\ & \frac{9139}{14515200} - \frac{4276401149}{958003200}\xi + \frac{1317793}{38320128}\xi^2 - \frac{2883059}{43545600}\xi^3 + \frac{4}{15}\xi^5 - \frac{962}{1575}\xi^6 + \frac{698}{945}\xi^7 \\
& + \frac{460}{1701}\xi^9 - \frac{1168}{14175}\xi^{10} + \frac{64}{4455}\xi^{11} - \frac{512}{467775}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_{1/2} =\ & \frac{23773}{5806080} + \frac{24743560769}{1916006400}\xi + \frac{103393811}{1916006400}\xi^2 - \frac{517}{460800}\xi^3 - \frac{7}{15}\xi^5 + \frac{69}{50}\xi^6 - \frac{18353}{9450}\xi^7 \\
& + \frac{1432}{1701}\xi^9 - \frac{3824}{14175}\xi^{10} + \frac{1088}{22275}\xi^{11} - \frac{256}{66825}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_{3/4} =\ & \frac{23069}{2903040} - \frac{1901180509}{87091200}\xi + \frac{85080211}{958003200}\xi^2 - \frac{3925969}{43545600}\xi^3 + \frac{25}{45}\xi^5 - \frac{4006}{2025}\xi^6 + \frac{1594}{525}\xi^7 \\
& + \frac{284}{189}\xi^9 - \frac{2384}{4725}\xi^{10} + \frac{64}{675}\xi^{11} - \frac{512}{66825}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_1 =\ & \frac{15953}{1658880} + \frac{17788306631}{766402560}\xi + \frac{7434899}{109486080}\xi^2 + \frac{575849}{34836480}\xi^3 - \frac{7}{12}\xi^5 + \frac{691}{360}\xi^6 - \frac{2914}{945}\xi^7 \\
& - \frac{2864}{1701}\xi^9 + \frac{1672}{2835}\xi^{10} - \frac{512}{4455}\xi^{11} + \frac{128}{13365}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_{5/4} =\ & \frac{18617}{2903040} - \frac{15515774387}{958003200}\xi + \frac{51779383}{958003200}\xi^2 - \frac{232117}{4838400}\xi^3 + \frac{28}{75}\xi^5 - \frac{94}{75}\xi^6 + \frac{9782}{4725}\xi^7 \\
& + \frac{10324}{8505}\xi^9 - \frac{6256}{14175}\xi^{10} + \frac{1984}{22275}\xi^{11} - \frac{512}{66825}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_{3/2} =\ & \frac{9491}{4147200} + \frac{13614927413}{1916006400}\xi + \frac{5442979}{383201280}\xi^2 + \frac{79309}{12441600}\xi^3 - \frac{7}{45}\xi^5 + \frac{2143}{4050}\xi^6 - \frac{187}{210}\xi^7 \\
& - \frac{104}{189}\xi^9 + \frac{976}{4725}\xi^{10} - \frac{64}{1485}\xi^{11} + \frac{256}{66825}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_{7/4} =\ & \frac{151}{580608} - \frac{1740243377}{958003200}\xi + \frac{2431837}{958003200}\xi^2 - \frac{162431}{43545600}\xi^3 + \frac{4}{105}\xi^5 - \frac{206}{1575}\xi^6 + \frac{1054}{4725}\xi^7 \\
& + \frac{244}{1701}\xi^9 - \frac{112}{2025}\xi^{10} + \frac{1856}{155925}\xi^{11} - \frac{512}{467775}\xi^{12},
\end{aligned}$$

$$\begin{aligned}
\beta_2 =\ & \frac{5}{4644864} + \frac{1568469677}{7664025600}\xi - \frac{468073}{7664025600}\xi^2 + \frac{12683}{38707200}\xi^3 - \frac{1}{240}\xi^5 + \frac{121}{8400}\xi^6 - \frac{67}{2700}\xi^7 \\
& - \frac{4}{243}\xi^9 + \frac{92}{14175}\xi^{10} - \frac{32}{22275}\xi^{11} + \frac{64}{467775}\xi^{12}.
\end{aligned}$$

The block algorithm (5) is expanded to yield

$$
\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \tfrac{1}{4}hy'_n + \tfrac{1}{2!}\left(\tfrac{1}{4}h\right)^2 y''_n + \tfrac{1}{3!}\left(\tfrac{1}{4}h\right)^3 y'''_n + h^4\Big(\psi_{011}f_n + \psi_{012}f_{n+\frac{1}{4}} + \psi_{013}f_{n+\frac{1}{2}} + \psi_{014}f_{n+\frac{3}{4}} + \psi_{015}f_{n+1} \\
&\quad + \psi_{016}f_{n+\frac{5}{4}} + \psi_{017}f_{n+\frac{3}{2}} + \psi_{018}f_{n+\frac{7}{4}} + \psi_{019}f_{n+2}\Big), \\
y_{n+\frac{1}{2}} &= y_n + \tfrac{1}{2}hy'_n + \tfrac{1}{2!}\left(\tfrac{1}{2}h\right)^2 y''_n + \tfrac{1}{3!}\left(\tfrac{1}{2}h\right)^3 y'''_n + h^4\Big(\psi_{021}f_n + \psi_{022}f_{n+\frac{1}{4}} + \psi_{023}f_{n+\frac{1}{2}} + \psi_{024}f_{n+\frac{3}{4}} + \psi_{025}f_{n+1} \\
&\quad + \psi_{026}f_{n+\frac{5}{4}} + \psi_{027}f_{n+\frac{3}{2}} + \psi_{028}f_{n+\frac{7}{4}} + \psi_{029}f_{n+2}\Big), \\
y_{n+\frac{3}{4}} &= y_n + \tfrac{3}{4}hy'_n + \tfrac{1}{2!}\left(\tfrac{3}{4}h\right)^2 y''_n + \tfrac{1}{3!}\left(\tfrac{3}{4}h\right)^3 y'''_n + h^4\Big(\psi_{031}f_n + \psi_{032}f_{n+\frac{1}{4}} + \psi_{033}f_{n+\frac{1}{2}} + \psi_{034}f_{n+\frac{3}{4}} + \psi_{035}f_{n+1} \\
&\quad + \psi_{036}f_{n+\frac{5}{4}} + \psi_{037}f_{n+\frac{3}{2}} + \psi_{038}f_{n+\frac{7}{4}} + \psi_{039}f_{n+2}\Big), \\
y_{n+1} &= y_n + hy'_n + \tfrac{1}{2!}h^2 y''_n + \tfrac{1}{3!}h^3 y'''_n + h^4\Big(\psi_{041}f_n + \psi_{042}f_{n+\frac{1}{4}} + \psi_{043}f_{n+\frac{1}{2}} + \psi_{044}f_{n+\frac{3}{4}} + \psi_{045}f_{n+1} \\
&\quad + \psi_{046}f_{n+\frac{5}{4}} + \psi_{047}f_{n+\frac{3}{2}} + \psi_{048}f_{n+\frac{7}{4}} + \psi_{049}f_{n+2}\Big), \\
y_{n+\frac{5}{4}} &= y_n + \tfrac{5}{4}hy'_n + \tfrac{1}{2!}\left(\tfrac{5}{4}h\right)^2 y''_n + \tfrac{1}{3!}\left(\tfrac{5}{4}h\right)^3 y'''_n + h^4\Big(\psi_{051}f_n + \psi_{052}f_{n+\frac{1}{4}} + \psi_{053}f_{n+\frac{1}{2}} + \psi_{054}f_{n+\frac{3}{4}} + \psi_{055}f_{n+1} \\
&\quad + \psi_{056}f_{n+\frac{5}{4}} + \psi_{057}f_{n+\frac{3}{2}} + \psi_{058}f_{n+\frac{7}{4}} + \psi_{059}f_{n+2}\Big), \\
y_{n+\frac{3}{2}} &= y_n + \tfrac{3}{2}hy'_n + \tfrac{1}{2!}\left(\tfrac{3}{2}h\right)^2 y''_n + \tfrac{1}{3!}\left(\tfrac{3}{2}h\right)^3 y'''_n + h^4\Big(\psi_{061}f_n + \psi_{062}f_{n+\frac{1}{4}} + \psi_{063}f_{n+\frac{1}{2}} + \psi_{064}f_{n+\frac{3}{4}} + \psi_{065}f_{n+1} \\
&\quad + \psi_{066}f_{n+\frac{5}{4}} + \psi_{067}f_{n+\frac{3}{2}} + \psi_{068}f_{n+\frac{7}{4}} + \psi_{069}f_{n+2}\Big), \\
y_{n+\frac{7}{4}} &= y_n + \tfrac{7}{4}hy'_n + \tfrac{1}{2!}\left(\tfrac{7}{4}h\right)^2 y''_n + \tfrac{1}{3!}\left(\tfrac{7}{4}h\right)^3 y'''_n + h^4\Big(\psi_{071}f_n + \psi_{072}f_{n+\frac{1}{4}} + \psi_{073}f_{n+\frac{1}{2}} + \psi_{074}f_{n+\frac{3}{4}} + \psi_{075}f_{n+1} \\
&\quad + \psi_{076}f_{n+\frac{5}{4}} + \psi_{077}f_{n+\frac{3}{2}} + \psi_{078}f_{n+\frac{7}{4}} + \psi_{079}f_{n+2}\Big), \\
y_{n+2} &= y_n + 2hy'_n + \tfrac{1}{2!}(2h)^2 y''_n + \tfrac{1}{3!}(2h)^3 y'''_n + h^4\Big(\psi_{081}f_n + \psi_{082}f_{n+\frac{1}{4}} + \psi_{083}f_{n+\frac{1}{2}} + \psi_{084}f_{n+\frac{3}{4}} + \psi_{085}f_{n+1} \\
&\quad + \psi_{086}f_{n+\frac{5}{4}} + \psi_{087}f_{n+\frac{3}{2}} + \psi_{088}f_{n+\frac{7}{4}} + \psi_{089}f_{n+2}\Big).
\end{aligned} \tag{8}
$$

Likewise, expanding Eq. (6), we obtain:

$$
\begin{aligned}
y'_{n+\frac{1}{4}} &= y'_n + \tfrac{1}{4}hy''_n + \frac{(\frac{1}{4}h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{111}f_n + \Omega_{112}f_{n+\frac{1}{4}} + \Omega_{113}f_{n+\frac{1}{2}} + \Omega_{114}f_{n+\frac{3}{4}} + \Omega_{115}f_{n+1} + \Omega_{116}f_{n+\frac{5}{4}} + \Omega_{117}f_{n+\frac{3}{2}} + \Omega_{118}f_{n+\frac{7}{4}} + \Omega_{119}f_{n+2}\right), \\
y'_{n+\frac{1}{2}} &= y'_n + \tfrac{1}{2}hy''_n + \frac{(\frac{1}{2}h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{121}f_n + \Omega_{122}f_{n+\frac{1}{4}} + \Omega_{123}f_{n+\frac{1}{2}} + \Omega_{124}f_{n+\frac{3}{4}} + \Omega_{125}f_{n+1} + \Omega_{126}f_{n+\frac{5}{4}} + \Omega_{127}f_{n+\frac{3}{2}} + \Omega_{128}f_{n+\frac{7}{4}} + \Omega_{129}f_{n+2}\right), \\
y'_{n+\frac{3}{4}} &= y'_n + \tfrac{3}{4}hy''_n + \frac{(\frac{3}{4}h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{131}f_n + \Omega_{132}f_{n+\frac{1}{4}} + \Omega_{133}f_{n+\frac{1}{2}} + \Omega_{134}f_{n+\frac{3}{4}} + \Omega_{135}f_{n+1} + \Omega_{136}f_{n+\frac{5}{4}} + \Omega_{137}f_{n+\frac{3}{2}} + \Omega_{138}f_{n+\frac{7}{4}} + \Omega_{139}f_{n+2}\right),
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + hy''_n + \tfrac{h^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{141}f_n + \Omega_{142}f_{n+\frac{1}{4}} + \Omega_{143}f_{n+\frac{1}{2}} + \Omega_{144}f_{n+\frac{3}{4}} + \Omega_{145}f_{n+1} + \Omega_{146}f_{n+\frac{5}{4}} + \Omega_{147}f_{n+\frac{3}{2}} + \Omega_{148}f_{n+\frac{7}{4}} + \Omega_{149}f_{n+2}\right), \\
y'_{n+\frac{5}{4}} &= y'_n + \tfrac{5}{4}hy''_n + \tfrac{(\frac{5}{4}h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{151}f_n + \Omega_{152}f_{n+\frac{1}{4}} + \Omega_{153}f_{n+\frac{1}{2}} + \Omega_{154}f_{n+\frac{3}{4}} + \Omega_{155}f_{n+1} + \Omega_{156}f_{n+\frac{5}{4}} + \Omega_{157}f_{n+\frac{3}{2}} + \Omega_{158}f_{n+\frac{7}{4}} + \Omega_{159}f_{n+2}\right), \\
y'_{n+\frac{3}{2}} &= y'_n + \tfrac{3}{2}hy''_n + \tfrac{(\frac{3}{2}h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{161}f_n + \Omega_{162}f_{n+\frac{1}{4}} + \Omega_{163}f_{n+\frac{1}{2}} + \Omega_{164}f_{n+\frac{3}{4}} + \Omega_{165}f_{n+1} + \Omega_{166}f_{n+\frac{5}{4}} + \Omega_{167}f_{n+\frac{3}{2}} + \Omega_{168}f_{n+\frac{7}{4}} + \Omega_{169}f_{n+2}\right), \\
y'_{n+\frac{7}{4}} &= y'_n + \tfrac{7}{4}hy''_n + \tfrac{(\frac{7}{4}h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{171}f_n + \Omega_{172}f_{n+\frac{1}{4}} + \Omega_{173}f_{n+\frac{1}{2}} + \Omega_{174}f_{n+\frac{3}{4}} + \Omega_{175}f_{n+1} + \Omega_{176}f_{n+\frac{5}{4}} + \Omega_{177}f_{n+\frac{3}{2}} + \Omega_{178}f_{n+\frac{7}{4}} + \Omega_{179}f_{n+2}\right), \\
y'_{n+2} &= y'_n + 2hy''_n + \tfrac{(2h)^2}{2!}y'''_n \\
&\quad + h^3\left(\Omega_{181}f_n + \Omega_{182}f_{n+\frac{1}{4}} + \Omega_{183}f_{n+\frac{1}{2}} + \Omega_{184}f_{n+\frac{3}{4}} + \Omega_{185}f_{n+1} + \Omega_{186}f_{n+\frac{5}{4}} + \Omega_{187}f_{n+\frac{3}{2}} + \Omega_{188}f_{n+\frac{7}{4}} + \Omega_{189}f_{n+2}\right).
\end{aligned}
$$

$$
\begin{aligned}
y''_{n+\frac{1}{4}} &= y''_n + \tfrac{1}{4}hy'''_n + h^2(\Omega_{211}f_n + \Omega_{212}f_{n+\frac{1}{4}} + \Omega_{213}f_{n+\frac{1}{2}} + \Omega_{214}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{215}f_{n+1} + \Omega_{216}f_{n+\frac{5}{4}} + \Omega_{217}f_{n+\frac{3}{2}} + \Omega_{218}f_{n+\frac{7}{4}} + \Omega_{219}f_{n+2}), \\
y''_{n+\frac{1}{2}} &= y''_n + \tfrac{1}{2}hy'''_n + h^2(\Omega_{221}f_n + \Omega_{222}f_{n+\frac{1}{4}} + \Omega_{223}f_{n+\frac{1}{2}} + \Omega_{224}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{225}f_{n+1} + \Omega_{226}f_{n+\frac{5}{4}} + \Omega_{227}f_{n+\frac{3}{2}} + \Omega_{228}f_{n+\frac{7}{4}} + \Omega_{229}f_{n+2}), \\
y''_{n+\frac{3}{4}} &= y''_n + \tfrac{3}{4}hy'''_n + h^2(\Omega_{231}f_n + \Omega_{232}f_{n+\frac{1}{4}} + \Omega_{233}f_{n+\frac{1}{2}} + \Omega_{234}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{235}f_{n+1} + \Omega_{236}f_{n+\frac{5}{4}} + \Omega_{237}f_{n+\frac{3}{2}} + \Omega_{238}f_{n+\frac{7}{4}} + \Omega_{239}f_{n+2}), \\
y''_{n+1} &= y''_n + hy'''_n + h^2(\Omega_{241}f_n + \Omega_{242}f_{n+\frac{1}{4}} + \Omega_{243}f_{n+\frac{1}{2}} + \Omega_{244}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{245}f_{n+1} + \Omega_{246}f_{n+\frac{5}{4}} + \Omega_{247}f_{n+\frac{3}{2}} + \Omega_{248}f_{n+\frac{7}{4}} + \Omega_{249}f_{n+2}), \\
y''_{n+\frac{5}{4}} &= y''_n + \tfrac{5}{4}hy'''_n + h^2(\Omega_{251}f_n + \Omega_{252}f_{n+\frac{1}{4}} + \Omega_{253}f_{n+\frac{1}{2}} + \Omega_{254}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{255}f_{n+1} + \Omega_{256}f_{n+\frac{5}{4}} + \Omega_{257}f_{n+\frac{3}{2}} + \Omega_{258}f_{n+\frac{7}{4}} + \Omega_{259}f_{n+2}), \\
y''_{n+\frac{3}{2}} &= y''_n + \tfrac{3}{2}hy'''_n + h^2(\Omega_{261}f_n + \Omega_{262}f_{n+\frac{1}{4}} + \Omega_{263}f_{n+\frac{1}{2}} + \Omega_{264}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{265}f_{n+1} + \Omega_{266}f_{n+\frac{5}{4}} + \Omega_{267}f_{n+\frac{3}{2}} + \Omega_{268}f_{n+\frac{7}{4}} + \Omega_{269}f_{n+2}), \\
y''_{n+\frac{7}{4}} &= y''_n + \tfrac{7}{4}hy'''_n + h^2(\Omega_{271}f_n + \Omega_{272}f_{n+\frac{1}{4}} + \Omega_{273}f_{n+\frac{1}{2}} + \Omega_{274}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{275}f_{n+1} + \Omega_{276}f_{n+\frac{5}{4}} + \Omega_{277}f_{n+\frac{3}{2}} + \Omega_{278}f_{n+\frac{7}{4}} + \Omega_{279}f_{n+2}), \\
y''_{n+2} &= y''_n + 2hy'''_n + h^2(\Omega_{281}f_n + \Omega_{282}f_{n+\frac{1}{4}} + \Omega_{283}f_{n+\frac{1}{2}} + \Omega_{284}f_{n+\frac{3}{4}} \\
&\quad + \Omega_{285}f_{n+1} + \Omega_{286}f_{n+\frac{5}{4}} + \Omega_{287}f_{n+\frac{3}{2}} + \Omega_{288}f_{n+\frac{7}{4}} + \Omega_{289}f_{n+2}).
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{4}} &= y'''_n + h\left(\Omega_{311}f_n + \Omega_{312}f_{n+\frac{1}{4}} + \Omega_{313}f_{n+\frac{1}{2}} + \Omega_{314}f_{n+\frac{3}{4}} + \Omega_{315}f_{n+1}\right. \\
&\quad \left. + \Omega_{316}f_{n+\frac{5}{4}} + \Omega_{317}f_{n+\frac{3}{2}} + \Omega_{318}f_{n+\frac{7}{4}} + \Omega_{319}f_{n+2}\right),
\end{aligned}
$$

$$
\begin{aligned}
y'''_{n+\frac{1}{2}} &= y'''_n + h\Big(\Omega_{321} f_n + \Omega_{322} f_{n+\frac{1}{4}} + \Omega_{323} f_{n+\frac{1}{2}} + \Omega_{324} f_{n+\frac{3}{4}} + \Omega_{325} f_{n+1} \\
&\quad + \Omega_{326} f_{n+\frac{5}{4}} + \Omega_{327} f_{n+\frac{3}{2}} + \Omega_{328} f_{n+\frac{7}{4}} + \Omega_{329} f_{n+2}\Big), \\
y'''_{n+\frac{3}{4}} &= y'''_n + h\Big(\Omega_{331} f_n + \Omega_{332} f_{n+\frac{1}{4}} + \Omega_{333} f_{n+\frac{1}{2}} + \Omega_{334} f_{n+\frac{3}{4}} + \Omega_{335} f_{n+1} \\
&\quad + \Omega_{336} f_{n+\frac{5}{4}} + \Omega_{337} f_{n+\frac{3}{2}} + \Omega_{338} f_{n+\frac{7}{4}} + \Omega_{339} f_{n+2}\Big), \\
y'''_{n+1} &= y'''_n + h\Big(\Omega_{341} f_n + \Omega_{342} f_{n+\frac{1}{4}} + \Omega_{343} f_{n+\frac{1}{2}} + \Omega_{344} f_{n+\frac{3}{4}} + \Omega_{345} f_{n+1} \\
&\quad + \Omega_{346} f_{n+\frac{5}{4}} + \Omega_{347} f_{n+\frac{3}{2}} + \Omega_{348} f_{n+\frac{7}{4}} + \Omega_{349} f_{n+2}\Big), \\
y'''_{n+\frac{5}{4}} &= y'''_n + h\Big(\Omega_{351} f_n + \Omega_{352} f_{n+\frac{1}{4}} + \Omega_{353} f_{n+\frac{1}{2}} + \Omega_{354} f_{n+\frac{3}{4}} + \Omega_{355} f_{n+1} \\
&\quad + \Omega_{356} f_{n+\frac{5}{4}} + \Omega_{357} f_{n+\frac{3}{2}} + \Omega_{358} f_{n+\frac{7}{4}} + \Omega_{359} f_{n+2}\Big), \\
y'''_{n+\frac{3}{2}} &= y'''_n + h\Big(\Omega_{361} f_n + \Omega_{362} f_{n+\frac{1}{4}} + \Omega_{363} f_{n+\frac{1}{2}} + \Omega_{364} f_{n+\frac{3}{4}} + \Omega_{365} f_{n+1} \\
&\quad + \Omega_{366} f_{n+\frac{5}{4}} + \Omega_{367} f_{n+\frac{3}{2}} + \Omega_{368} f_{n+\frac{7}{4}} + \Omega_{369} f_{n+2}\Big), \\
y'''_{n+\frac{7}{4}} &= y'''_n + h\Big(\Omega_{371} f_n + \Omega_{372} f_{n+\frac{1}{4}} + \Omega_{373} f_{n+\frac{1}{2}} + \Omega_{374} f_{n+\frac{3}{4}} + \Omega_{375} f_{n+1} \\
&\quad + \Omega_{376} f_{n+\frac{5}{4}} + \Omega_{377} f_{n+\frac{3}{2}} + \Omega_{378} f_{n+\frac{7}{4}} + \Omega_{379} f_{n+2}\Big), \\
y'''_{n+2} &= y'''_n + h\Big(\Omega_{381} f_n + \Omega_{382} f_{n+\frac{1}{4}} + \Omega_{318} f_{n+\frac{1}{2}} + \Omega_{384} f_{n+\frac{3}{4}} + \Omega_{385} f_{n+1} \\
&\quad + \Omega_{386} f_{n+\frac{5}{4}} + \Omega_{387} f_{n+\frac{3}{2}} + \Omega_{388} f_{n+\frac{7}{4}} + \Omega_{389} f_{n+2}\Big).
\end{aligned} \tag{11}
$$

Hence, we determine the unknown coefficients of ψ and Ω by solving $\psi_{\xi j} = U^{-1}G$ and $\Omega_{\xi j \varsigma} = U^{-1}D$.

3. Numerical scheme of the proposed method

The sufficient and necessary conditions for analysis were basically scrutinized in this section.

3.1. Order and error constant

We consider the linear operator $L\left[y\left(t_n\right);h\right]$, we use Corollaries 2 and 3 below, to determine the order and error constant of the proposed method.

Corollary 2

According to Ref. [3], the linear operator, $L\left[y\left(t_n\right);h\right]$ associated with the local truncation error of the proposed method is $C_{07}h^{07}y^{07}\left(t_n\right)+0\left(h^{11}\right)$.

Proof

As stated in Ref. [3], the linear difference operators corresponding to the proposed method are expressed as:

$$
\begin{aligned}
L\left[y(t_n);h\right] = y\left(t_n + \tfrac{1}{4}h\right) - \Big(\alpha_{\frac{1}{4}}\left(t_n + \tfrac{1}{4}h\right) + \alpha_1\left(t_n + h\right) + \alpha_{\frac{3}{2}}\left(t_n + \tfrac{3}{2}h\right) \\
+ \alpha_2\left(x_n + 2h\right) + h^4 \sum_{i=0}^{\xi}\left(\beta_i(t) f_{n+i} + \beta_\xi(t) f_{n+\xi}\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
L\left[y(t_n);h\right] = y\left(t_n + \tfrac{1}{2}h\right) - \Big(\alpha_{\frac{1}{4}}\left(t_n + \tfrac{1}{4}h\right) + \alpha_1\left(t_n + h\right) + \alpha_{\frac{3}{2}}\left(t_n + \tfrac{3}{2}h\right) \\
+ \alpha_2\left(x_n + 2h\right) + h^4 \sum_{i=0}^{\xi}\left(\beta_i(t) f_{n+i} + \beta_\xi(t) f_{n+\xi}\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
L\left[y(t_n);h\right] &= y\left(t_n+\tfrac{3}{4}h\right)-\Big(\alpha_{\frac{1}{4}}\left(t_n+\tfrac{1}{4}h\right)+\alpha_1\left(t_n+h\right)+\alpha_{\frac{3}{2}}\left(t_n+\tfrac{3}{2}h\right)\\
&\quad+\alpha_2\left(x_n+2h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\Big),\\
L\left[y(t_n);h\right] &= y\left(t_n+h\right)-\Big(\alpha_{\frac{1}{4}}\left(t_n+\tfrac{1}{4}h\right)+\alpha_1\left(t_n+h\right)+\alpha_{\frac{3}{2}}\left(t_n+\tfrac{3}{2}h\right)\\
&\quad+\alpha_2\left(x_n+2h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\Big),\\
L\left[y(t_n);h\right] &= y\left(t_n+\tfrac{5}{4}h\right)-\Big(\alpha_{\frac{1}{4}}\left(t_n+\tfrac{1}{4}h\right)+\alpha_1\left(t_n+h\right)+\alpha_{\frac{3}{2}}\left(t_n+\tfrac{3}{2}h\right)\\
&\quad+\alpha_2\left(x_n+2h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\Big),\\
L\left[y(t_n);h\right] &= y\left(t_n+\tfrac{3}{2}h\right)-\Big(\alpha_{\frac{1}{4}}\left(t_n+\tfrac{1}{4}h\right)+\alpha_1\left(t_n+h\right)+\alpha_{\frac{3}{2}}\left(t_n+\tfrac{3}{2}h\right)\\
&\quad+\alpha_2\left(x_n+2h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\Big),\\
L\left[y(t_n);h\right] &= y\left(t_n+\tfrac{7}{4}h\right)-\Big(\alpha_{\frac{1}{4}}\left(t_n+\tfrac{1}{4}h\right)+\alpha_1\left(t_n+h\right)+\alpha_{\frac{3}{2}}\left(t_n+\tfrac{3}{2}h\right)\\
&\quad+\alpha_2\left(x_n+2h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\Big),\\
L\left[y(t_n);h\right] &= y\left(t_n+2h\right)-\Big(\alpha_{\frac{1}{4}}\left(t_n+\tfrac{1}{4}h\right)+\alpha_1\left(t_n+h\right)+\alpha_{\frac{3}{2}}\left(t_n+\tfrac{3}{2}h\right)\\
&\quad+\alpha_2\left(x_n+2h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\Big).
\end{aligned}
\tag{12}
$$

Corollary 3

According to Ref. [3], the local truncation error of the new method assumes that $y(t)$ is sufficiently differentiable.By expanding $y(t_n+qh)$ and $y(t_n+jh)$ about t_n using a Taylor series, we obtain:

$$
\begin{aligned}
L_{\frac{1}{4}}\left[y(t_n);h\right] &= -2.2175\times10^{-7},\\
L_{\frac{1}{2}}\left[y(t_n);h\right] &= -3.2795\times10^{-6},\\
L_{\frac{3}{4}}\left[y(t_n);h\right] &= -1.3228\times10^{-5},\\
L_{1}\left[y(t_n);h\right] &= -3.3986\times10^{-5},\\
L_{\frac{5}{4}}\left[y(t_n);h\right] &= -6.9487\times10^{-5},\\
L_{\frac{3}{2}}\left[y(t_n);h\right] &= -1.2373\times10^{-4},\\
L_{\frac{7}{4}}\left[y(t_n);h\right] &= -1.9840\times10^{-4},\\
L_{2}\left[y(t_n);h\right] &= -2.8618\times10^{-4}.
\end{aligned}
$$

Proof

Using corollary 2, expand Eq. (12) and then simplify, we obtain:

$$
\begin{aligned}
L_{\frac{1}{4}}\left[y\left(t_n\right);h\right] &= \left(-2.2175\times 10^{-7}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{\frac{1}{2}}\left[y\left(t_n\right);h\right] &= \left(-3.2795\times 10^{-6}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{\frac{3}{4}}\left[y\left(t_n\right);h\right] &= \left(-1.3228\times 10^{-5}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{1}\left[y\left(t_n\right);h\right] &= \left(-3.3986\times 10^{-5}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{\frac{5}{4}}\left[y\left(t_n\right);h\right] &= \left(-6.9487\times 10^{-5}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{\frac{3}{2}}\left[y\left(t_n\right);h\right] &= \left(-1.2373\times 10^{-4}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{\frac{7}{4}}\left[y\left(t_n\right);h\right] &= \left(-1.9840\times 10^{-4}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right),\\
L_{2}\left[y\left(t_n\right);h\right] &= \left(-2.8618\times 10^{-4}\right)C_{07}h^{07}y^{(07)}\left(t_n\right)+O\left(h^{11}\right).
\end{aligned}
$$

3.2. Zero stability

A linear multistep method is said to be zero-stable for any well-behaved initial value problem if the roots of its characteristic equation $\rho(z) = 0$ lie within or on the unit circle in the complex plane, with any roots on the unit circle having multiplicity at most one.
Hence:

$$
\rho(z) = z^8 - \frac{1522}{35}z^7 + \frac{118124}{105}z^6 - \frac{102528}{5}z^5 + 273664z^4 - 2654208z^3 + 17891328z^2 + 7549742z + 150994944. \tag{13}
$$

Solving for z, we obtain $z = 1$, hence the method is zero stable.

3.3. Convergence

According to Ref. [1], a linear multistep method converges if it is consistent and zero-stable. Therefore, since the proposed method is consistent and zero stable, hence it is convergent.

3.4. Consistency

According to Refs [3, 5], a linear multistep method is said to be consistent if it has an order of convergence greater than or equal to zero. Thus, the proposed scheme is consistent.

3.5. Region of absolute stability

The complex values form the region of absolute stability of the proposed method, as the solution of the test problem, $y'''' = -\lambda^4 y$, remains bounded as $n \to \infty$.
The concept of A-stability, according to Ref. [3], is obtained by applying the test equation:

$$
y^{(k)} = \lambda^{(k)} y, \tag{14}
$$

gives

$$
Y_m = \mu(z)\, Y_{m-1}, z = \lambda h, \tag{15}
$$

where $\mu(z)$ is the amplification matrix of the form:

$$
\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^4\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^4\eta^{(1)}\right). \tag{16}
$$

The Eigenvalues, $(0,\ 0,\ \cdots,\ \xi_k)$ correspond to $\mu(z)$, where ξ_k is called the stability function and the stability function is given by:

$$
\zeta = -\frac{\begin{pmatrix} 131\,589567585z^8 - 3468\,986319486z^7 + 42210\,644799840z^6 \\ -412492\,607896852z^5 + 2664153\,236504256z^4 - 13438340\,522021184z^3 \\ -44356479\,052392192z^2 - 95879531\,652710400z + 94389581\,905920000 \end{pmatrix}}{\begin{pmatrix} 80\,015040000z^8 - 1739\,755584000z^7 + 22504\,039488000z^6 - 205094\,550528000z^5 \\ +1368577\,244160000z^4 - 6636767\,477760000z^3 \\ +22368364\,462080000z^2 - 47194790\,952960000z + 47194790\,952960000 \end{pmatrix}}.
$$

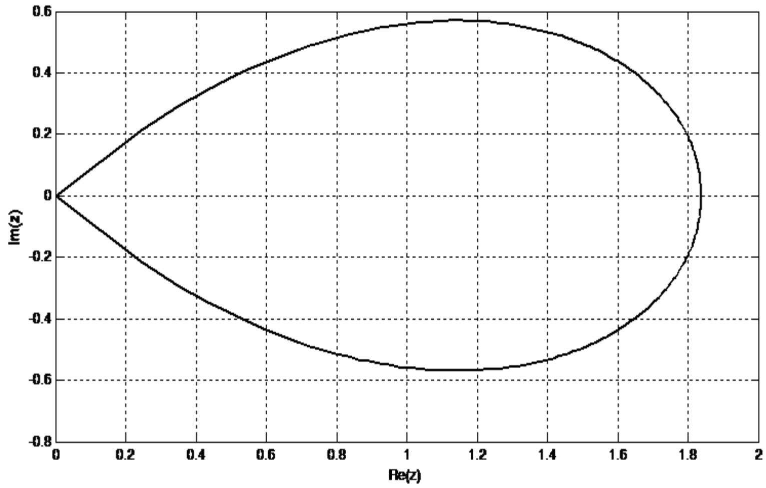

Figure 1. Showing an A-stable region of absolutely stability.

Using the boundary locus method on the new method, the stability polynomial is given as:

$$\begin{aligned}\overline{h}(w) = &\left(\frac{121}{526727577600}w^7 + \frac{1}{125411328}w^8\right)h^{16} + \left(-\frac{601}{1881169920}w^8 - \frac{4466603}{23702740992000}w^7\right)h^{14}\\ &+\left(-\frac{37954117}{8888527872000}w^7 + \frac{1237}{156764160}w^8\right)h^{12} + \left(-\frac{3137}{22394880}w^8 - \frac{37381}{373248000}w^7\right)h^{10}\\ &+\left(-\frac{5849399}{3919104000}w^7 + \frac{4123}{2239488}w^8\right)h^8 + \left(-\frac{367}{20736}w^8 - \frac{6067}{403200}w^7\right)h^6\\ &+\left(-\frac{121157}{907200}w^7 + \frac{1231}{10368}w^7\right)h^4 + \left(-\frac{1}{2}w^7 - \frac{1}{2}w^8\right)h^2 - 2w^7 + w^8.\end{aligned} \tag{17}$$

From the stability polynomial given by Eq. (17), the region of absolute stability is shown in Figure 1 as

4. Experimental problems and discussion

In this section, the accuracy and effectiveness of the proposed method (PM) are demonstrated through various higher-order initial value problems. These include second, third, and fourth-order OSDEs derived from physical problems, as well as linear and nonlinear systems, represented by equations (2), (3), and (4). All simulations were performed using the Maple 18 software package.

$$\text{Error} = \text{Absolute}(\text{Exact} - \text{Approximate}).$$

Problem 1: Consider the second-order oscillatory real-life problem: Simple Harmonic Motion. An object stretches a spring by 6 inches in equilibrium. Formulate the equation of motion and determine its general solution. Find the displacement of the object for $t > 0$, given that it is initially displaced 18 inches above equilibrium and has an initial downward velocity of $3\frac{ft}{s}$. Using second law of motion, gives

$$my''(t) + cy'(t) + ky(t) = F. \tag{18}$$

Setting $c = 0$ and$F = 0$, get

$$my''(t) + ky = 0 \Rightarrow y''(t) + \frac{k}{m}y(t) = 0. \tag{19}$$

The equation for the weight of the object is:

$$mg = k\Delta l \Rightarrow \frac{k}{m} = \frac{t}{\Delta l}, \tag{20}$$

substituting $t = 32\frac{ft}{s^2}, \Delta l = \frac{6}{12}ft$ into Eq. (18) to obtain:

$$\frac{k}{m} = \frac{32}{\frac{6}{12}} = 64. \tag{21}$$

Table 1. Numerical result for Problem 1 with that of OSHBM.

t	PM	Ref. [20]
0.1	5.94e-10	3.35e-07
0.2	8.23e-10	1.64e-06
0.3	2.94e-10	3.27e-06
0.4	4.05e-10	3.60e-06
0.5	1.77e-09	1.36e-06
0.6	2.06e-09	2.91e-06
0.7	7.85e-10	6.72e-06
0.8	9.56e-10	7.06e-06
0.9	3.01e-09	2.65e-06
1.0	3.24e-09	4.61e-06

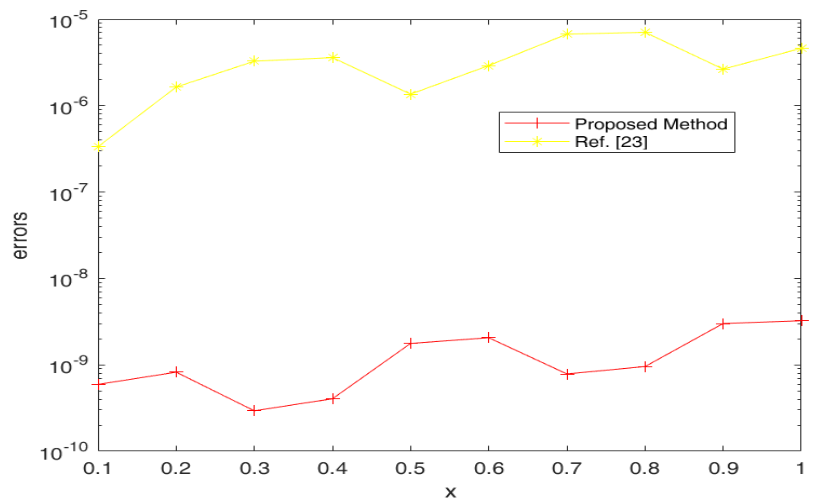

Figure 2. Comparison of the errors for Problem 1.

Now, Substitute Eq. (19) in Eq. (17), to get:

$$y''(t) + 64y = 0. \tag{22}$$

Expressing the displacement in feet, solving Eq. (20) subject to the initial downward velocity $y(0) = \frac{3}{2}$, $y'(0) = -3$ and $h = 0.1$:

$$y''(t) + 64y(t) = 0, \quad y(0) = \frac{3}{2}, \quad y'(0) = -3. \tag{23}$$

Exact solution is obtained as:

$$y(t) = -\frac{3}{8}\sin(8t) + \frac{3}{2}\cos(8t). \tag{24}$$

See Ref. [20].

Problem 2: Consider the highly non-stiff third order oscillatory problem

$$y'''(t) = 3\cos(t), \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2, \tag{25}$$

with the exact solution:

$$y(t) = t^2 - 3\sin(t) + 3t + 1. \tag{26}$$

Source: See Refs. [21, 22].

Problem 3

Consider the highly stiff system of fourth order oscillatory problem:

$$y^{iv} = \frac{-\left(8 + 25t + 30t^2 + 12t^3 + t^4\right)}{(1 + t^2)}, y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -3, \tag{27}$$

Table 2. Absolute errors for Problem 2 with that of TDHBM and SSBM.

t	PM	Ref. [21]	Ref. [22]
0.1	0	1.97e-16	0
0.2	0	1.26e-15	0
0.3	0	4.06e-15	6.00e-19
0.4	0	9.44e-15	1.70e-18
0.5	0	1.82e-14	3.70e-18
0.6	0	3.12e-14	6.80e-18
0.7	0	4.90e-14	1.13e-17
0.8	0	7.25e-14	1.73e-17
0.9	0	1.02e-13	2.49e-17
1.0	0	1.39e-13	3.45e-17

Table 3. Absolute errors for Problem 3 with that of MHPLMM and THBM.

t	PM	Ref. [23]	Ref. [24]
0.003125	2.4879e-14	2.4874e-14	1.9902e-14
0.003125	7.9657e-13	7.9720e-13	6.3793e-13
0.009375	6.0544e-12	6.3116e-14	4.8524e-12
0.001250	2.5538e-11	4.4102e-12	2.0482e-11
0.015625	7.8013e-11	5.7680e-12	6.2610e-11
0.018750	1.9431e-10	1.4918e-11	1.5605e-10
0.021875	4.2041e-10	9.1931e-11	3.3786e-10
0.025000	8.2046e-10	2.7786e-10	6.5982e-10
0.028125	1.4800e-09	6.4684e-10	1.1910e-09
0.031250	2.5088e-09	1.2977e-09	2.0204e-09

with exact solution:

$$y(t) = y\left(1 - t^2\right) exp(\ t). \tag{28}$$

Source: [23, 24].

5. Discussion of results

Table 1 presents the absolute errors for a second-order oscillatory real-life problem (Simple Harmonic Motion) using the proposed method and the results obtained in Ref. [20]. The values of the exact and computed solutions show the displacement of an object at different time intervals. The absolute errors (PM and Ref. [20]) provide insight into the accuracy of the numerical methods, with PM values consistently smaller than those obtained in Ref. [20] for most time steps, indicating that the proposed method offers higher precision. As time progresses, the errors fluctuate, as shown in Figure 2. However, both methods exhibit relatively small discrepancies when compared to the exact solution, highlighting the overall effectiveness of the proposed method for this oscillatory problem.

Table 2 presents the numerical results for a highly non-stiff third-order oscillatory problem (see Problem 2), comparing the proposed method with the existing methods in Refs. [21, 22]. The computed solutions align exactly with the exact solutions for all time steps, demonstrating that the proposed method produces highly accurate results compared to the results obtained in Refs. [21, 22]. The values for the reference methods show small discrepancies at each time step, indicating their precision, while the proposed method exhibits no detectable errors, suggesting it is highly effective for this particular oscillatory problem. Although both reference methods maintain low levels of error, the proposed method achieves optimal accuracy.

Table 3 presents the numerical results for a highly stiff fourth-order oscillatory problem, comparing the computed absolute errors for the proposed method with those from the existing methods in Refs. [23, 24]. The computed solutions closely match the exact solutions at each time step, with the proposed method showing very small absolute errors. The reference methods also show small discrepancies, but the proposed method consistently produces lower errors across all time steps. These results suggest that the proposed method performs well in handling highly stiff systems, demonstrating high precision and efficiency compared to the methods from Refs. [23, 24].

6. Summary and conclusion

This research article investigates the application of a higher-order linear block method for solving oscillatory problems. The study emphasizes the importance of numerical methods in handling complex differential equations that arise in various scientific and engineering fields. The linear block method is designed to offer an efficient approach for solving higher-order oscillatory differential equations, where traditional methods may struggle in terms of computational efficiency and accuracy. The study explores the theoretical foundations of the proposed block method, highlighting its key features, such as analysing the stability and convergence of the method. The accuracy of the proposed method was examined and compared with the existing methods, which demonstrate its robustness and effectiveness. The findings show that the proposed linear block method significantly reduces computational time and improves the accuracy of the solutions, making it a promising tool for solving complex oscillatory ordinary differential equations. The proposed method represents a significant advancement in the numerical solution of higher-order oscillatory ordinary differential equations. It offers notable improvements in terms of computational efficiency, accuracy, and stability compared to traditional methods. The method's ability to handle oscillatory behaviour effectively makes it suitable for a wide range of applications in science and engineering. Future research could explore further enhancements to the method and its application to more complex and real-world problems, ensuring its continued relevance in numerical analysis.

Data availability

This research did not generate or analyze any datasets. As such, data sharing is not applicable.

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