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Eigenfunctions and eigenvalues of modified double-center Woods-Saxon potential

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Abstract

This study investigates the quantum mechanical dynamics of nuclear fission in uranium-235 through the simulation of a modified double-center Woods-Saxon potential. The Schrödinger equation is solved using the Nikiforov-Uvarov method to obtain the eigenfunctions and eigenvalues characterizing the fission process. The research focuses on modeling binary fission to reveal subatomic dynamics such as energy transitions and particle probabilities during the splitting process. By solving the radial part of the three-dimensional Schrödinger equation, the eigenfunctions, eigenvalues, and probability densities of the system are computed. The study examines the transition of energy levels, eccentricity effects, and double-fold degeneracies associated with the fission process. Results demonstrate the dynamic evolution of fragment separation, from overlap at equilibrium to independent existence in distinct potential wells. The findings underscore the potential's capability to describe fission dynamics and provide insights into nuclear interactions and energy transitions in heavy nuclei. The results obtained showed that, the eigenfunctions exhibited even parity π^+ for eigenvalues; $n_z = 0,\ 2,\ 4, 6$ and odd parity π^- for $n_z = 1,\ 3,\ 5,\ 7$ with the number of nodes characterizing the state of the eigenfunctions. The energy eigenvalues were obtained by applying the continuity conditions at the boundary (z = 0) and solving the resultant transcendental equations at different values of eccentricity, z. The spectrum of eigenvalues showed double fold degeneracy as the transition from two independent schemes into one at infinite separation describes the complete asymmetric fission of ^{235}U into two fragments.

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Keywords: Modified Woods-Saxon potential, Schrödinger equation, Eigenfunctions, Nikiforov-Uvarov method

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1. Introduction

As a result of the energy crisis in the world today, scientists have explored different avenues of bridging the energy gap and one of these is nuclear fission. The idea behind nuclear fission is to generate nuclear energy for electricity generation, weapons' production and research. Nuclear reactors used in developed countries today work on the principle of nuclear fission to generate energy and accelerated elementary particles (i.e. protons and neutrons). Nuclear fission is a process whereby heavy nuclei split when they are bounded by light particle such as terminal nuclei or protons with the formation of lighter nuclei and the release of some elementals

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particles and energy. The terminal energy released is usually annexed for the generation of electricity. This energy is usually used to power industries and home across different countries of the world or power industries and homes across the countries where nuclear reaction is located [1]. The most common type of nuclear fission is binary fission. This involves the splitting of heavy nuclei into two elementary nuclei with the release of the light particles and energy [2]. For examples ^{235}U when bombarded by terminal neutrons splits into Barium (^{141}Ba), Krypton (^{92}Kr) and the release of neutrons and 177 MeV of energy [3]. Thus in this work, a modified double center Wood-Saxon potential model will be applied to simulate the process of binary fission of heavy nuclei and reveal its sub-atomic dynamics from the quantum mechanical perspective. Over the years, it is observed that, for the simulation of the nuclear fission process the use of two or more central forces is a more powerful tool than the one-center potential in the single particle shell model. This is because the sum of single particle energies rises to infinity for large deformations of one-center potentials as it contains a surface-energy term. However, the energies of two-center potential models do not go to infinity for large eccentricities because the surface does not go to infinity for large eccentricities [4]. As a result of this, so many researchers such as [4, 5, 7] etc. have attempted to describe different nuclear fission processes (binary and ternary) using the harmonic oscillator potential. This potential is not very realistic as it does not have a finite range and goes to infinity at long range (not well bound). The Wood-Saxon potential however, is smooth, finite, short-ranged and has adjustable parameters which fits and describes the nuclear fission model more appropriately. Thus in this work, a modified double centered Wood-Saxon potential model will be applied to the Schrödinger equation in other to simulate the process of binary fission in certain heavy nuclei by obtaining the eigenfunctions, eigenvalues and eccentricity of the two-body system for the asymmetric case of ^{235}U. The aim of this research paper is to simulate a modified double-center Wood-Saxon potential and apply it to describe the binary fission of ^{235}U.

2. Materials and methods

The materials used in this research work are personal computer with 4 GHz RAM, 500 GB HDD, Windows 10: core i5 and MATLAB software.

2.1. The radial part of the schrodinger equation

The radial part of the Schrodinger equation which contains the Woods-Saxon potential and the repulsive potential (centrifugal potential) is given by Nasirov *et al.* [8]:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{2\mu}{\mathrm{h}^2}\left(E_{n,l}+\ V_r-\frac{\mathrm{h}^2 l\,(l+1)}{2\mu r^2}\right)\right]R_{n,l}\,(r)=0 \qquad 0\le r\le\infty, \tag{1}$$

where $\mu=\frac{A_nA_U}{A_n+A_U}$ is the reduced mass, A_n and A_U are the masses of the neutron and ^{236}U, $E_{n,l}$ is the energy, $R_{n,l}\,(r)$ is the wavefunction and $n=0,1,2,3,\ldots$ and $l=0,1,2,3,\ldots$ are the energy levels and angular quantum number respectively, r is the distance from the center of the nucleus and $\hbar$ is the reduced Planck's constant and the term $\frac{l(l+1)}{r^2}$ is the centrifugal potential.

2.2. The Woods-Saxon potential

To solve the radial Schrödinger equation, the nuclear potential, V(r) is typically approximated as a mean-field potential that takes into account the average interaction of the nucleon with the other nucleons in the nucleus [9, 10]. This mean-field potential is often represented by a Woods-Saxon potential, given by:

$$V_{eff}\,(r)=V\,(r)+V_l(r), \tag{2}$$

where $V\,(r)$ is the modified Wood-Saxon potential and $V_l(r)$ is the angular momentum term.

$$V\,(r)=-\frac{V_o}{1+e^{\frac{r-R_o}{a_o}}}-\frac{w_o\,e^{\frac{r-R_o}{a_o}}}{\left(1+e^{\frac{r-R_o}{a_o}}\right)^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}, \tag{3}$$

where V_o and W_o *are* the depths of the potential well, R_o is the nuclear radius, a_o is the diffuseness parameter, r is the radial distance from the center of the nucleus, μ is the reduced mass of the neutron and core nucleus.

2.3. The modified double-center shell model using Wood-Saxon potential

The proposed collinear three-center shell modified Wood-Saxon potential model is illustrated viz:

$$V\,(r)=\begin{cases}-\frac{V_o}{1+e^{\left(\frac{r-R_o}{a_o}\right)}}-\frac{w_o\,e^{\left(\frac{r-R_o}{a_o}\right)}}{\left(1+e^{\left(\frac{r-R_o}{a_o}\right)}\right)^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}, & r>R_0\\ -\frac{V_o}{1+e^{-\left(\frac{r+R_o}{a_o}\right)}}-\frac{w_o\,e^{-\left(\frac{r+R_o}{a_o}\right)}}{\left(1+e^{-\left(\frac{r+R_o}{a_o}\right)}\right)^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}. & r<-R_0\end{cases} \tag{4}$$

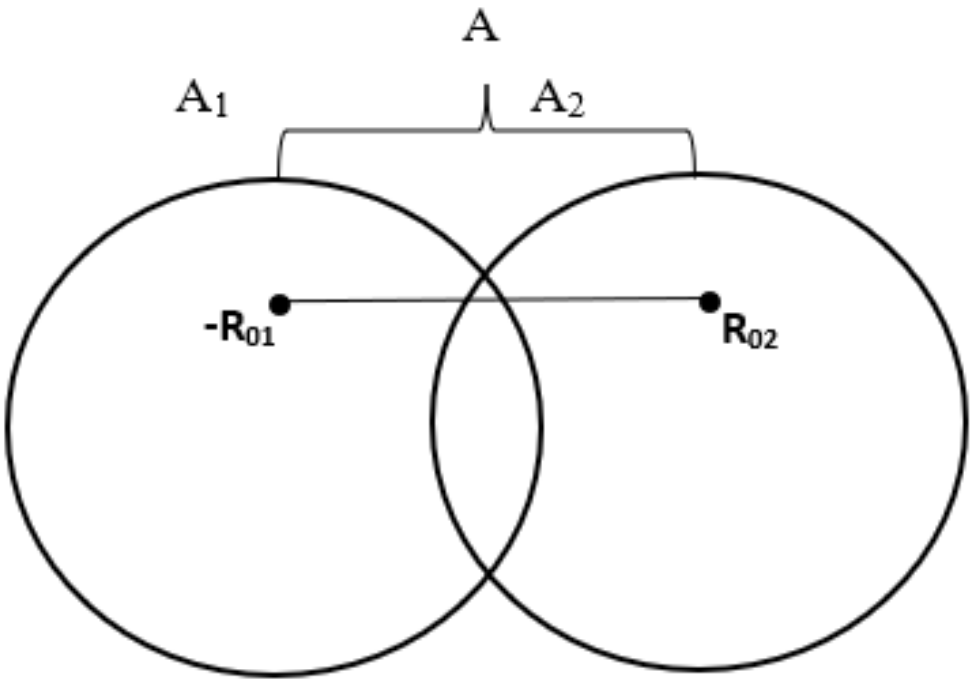

Figure 1. The double-center potential model.

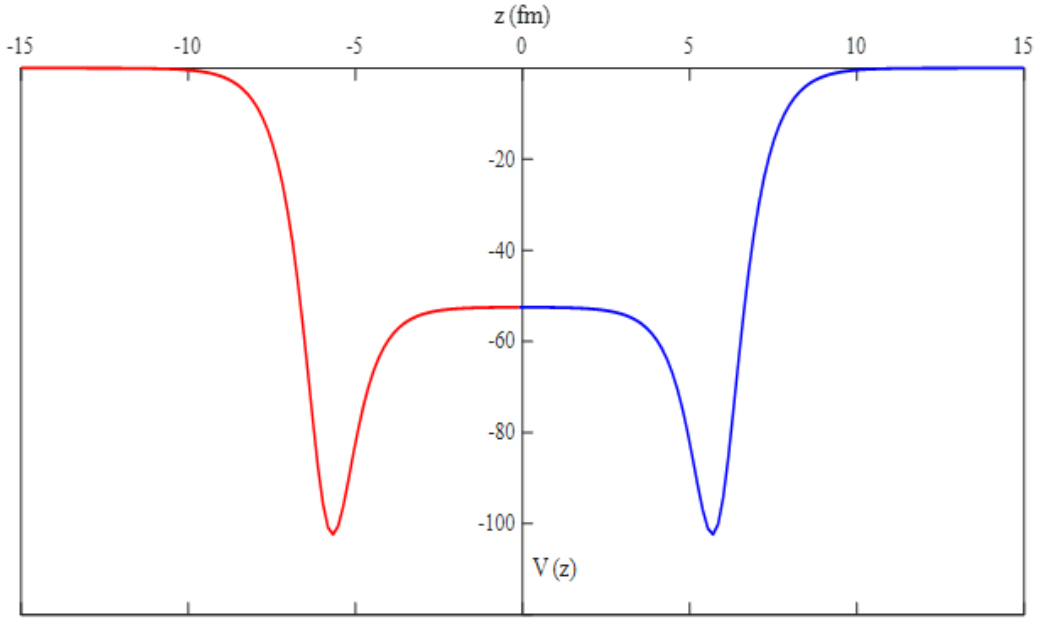

Figure 2. The modified double-centered Wood-Saxon potential well; $W_0 = 100MeV, 5 = 0.65\ fm,\ V_0 = 78.46\ MeV,\ R_0 = 3.5\ fm,\ \ l = 0.$

In this model, the molecule is divided into two centers (I & II) i.e. one to the left and one to the right. These two centers are assumed to be spherical, and their electron densities are represented by localized basis functions. These basis functions describe the distribution of electrons around each center and are often chosen to be Gaussian-type functions. The electron density of the whole system is then approximated as a linear combination of these basis functions, with each basis function centered at its corresponding center [11–13]. This allows for a mathematical representation of the electronic structure of the molecule based on the positions and types of these centers.

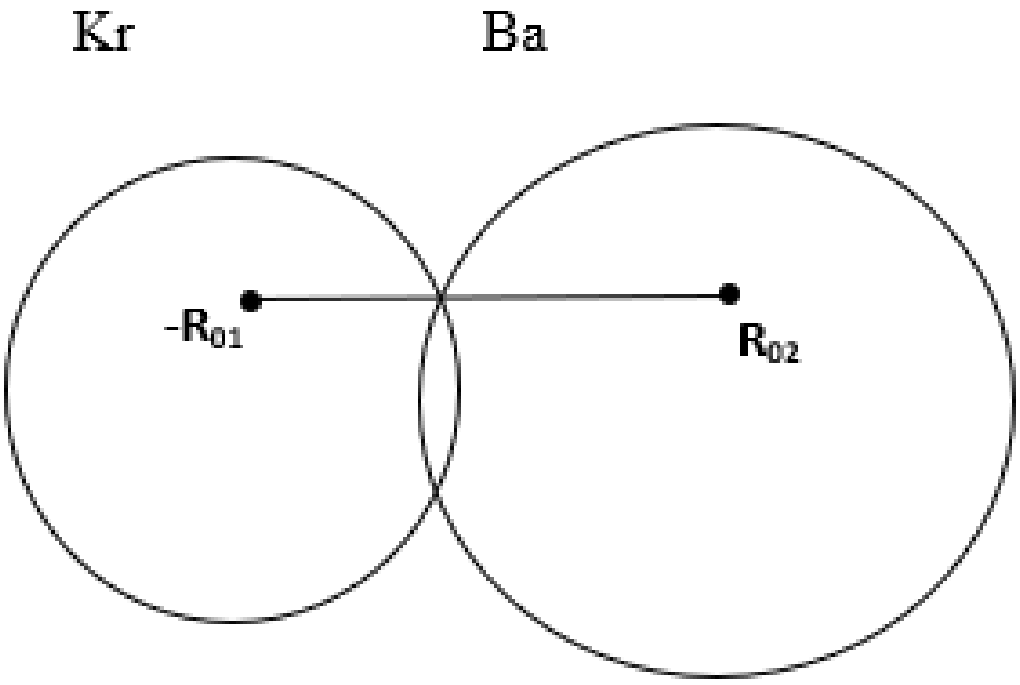

Figure 3. Major asymmetric fragments of the nuclear fission of ^{235}U.

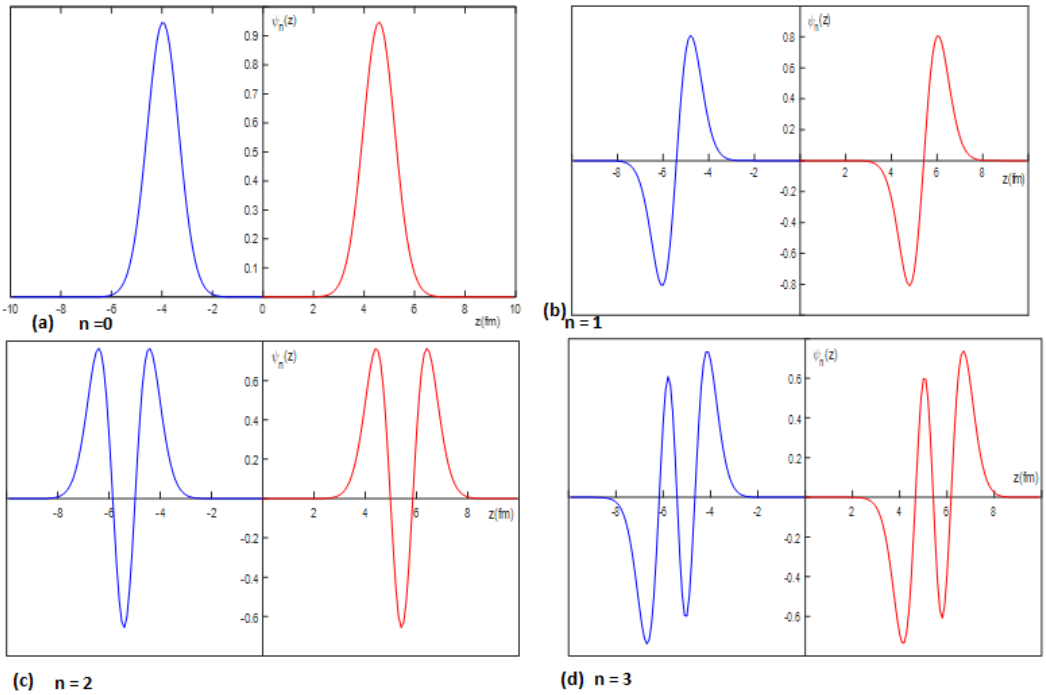

Figure 4. Radial eigenfunctions of the symmetric modified double-center Wood-Saxon potential for n = 0, 1, 2 & 3.

2.4. Wavefunctions of the Radial Schrodinger Equation with the Modified Wood Saxon Potential (MWSP)

A. Region I:

The eigenfunctions in region I are expressed in terms of Gauss hypergeometric function viz:

$$\psi_I(r) = \psi_{nl}(\beta) = \pm N'_{nl}\beta^{c_{12}}(1-C_3\beta)^{c_{13}}\ {}_2F_1(-\mathrm{n}, 1+C_{10}+C_{11}+n, C_{10}+1; C_3\beta) = 0. \quad (5)$$

B. Region II:

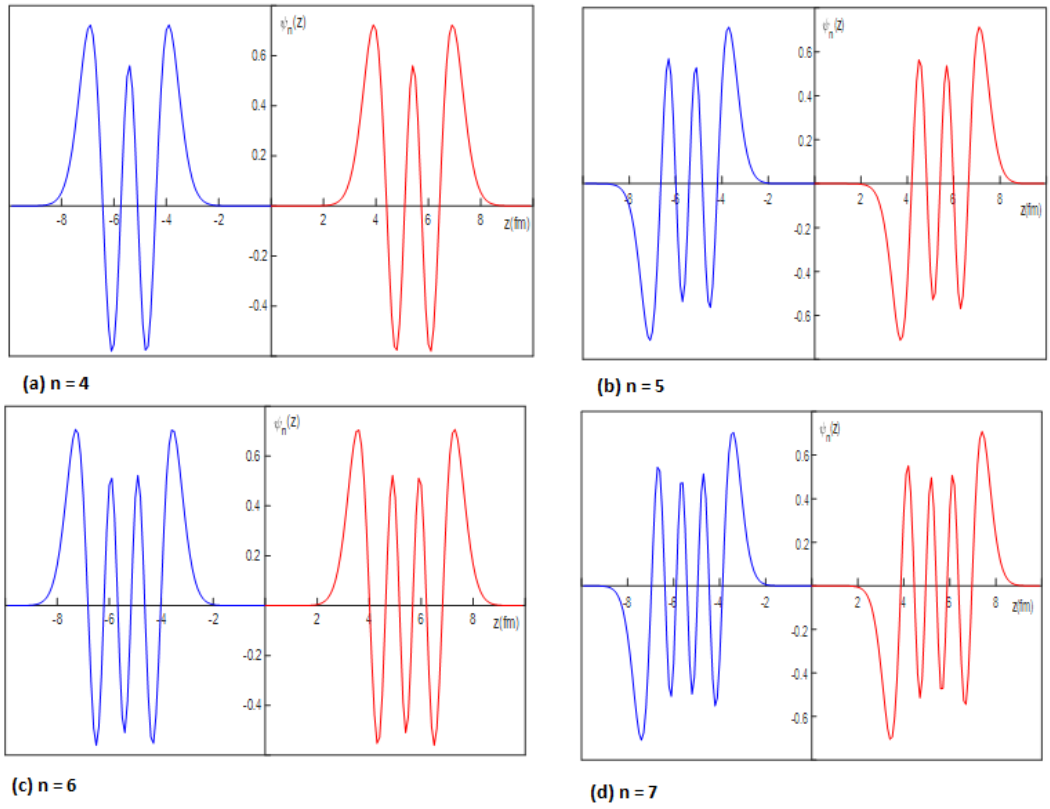

Figure 5. Radial eigenfunctions of the symmetric Modified Double-Center Wood-Saxon Potential for n = 4, 5, 6 & 7.

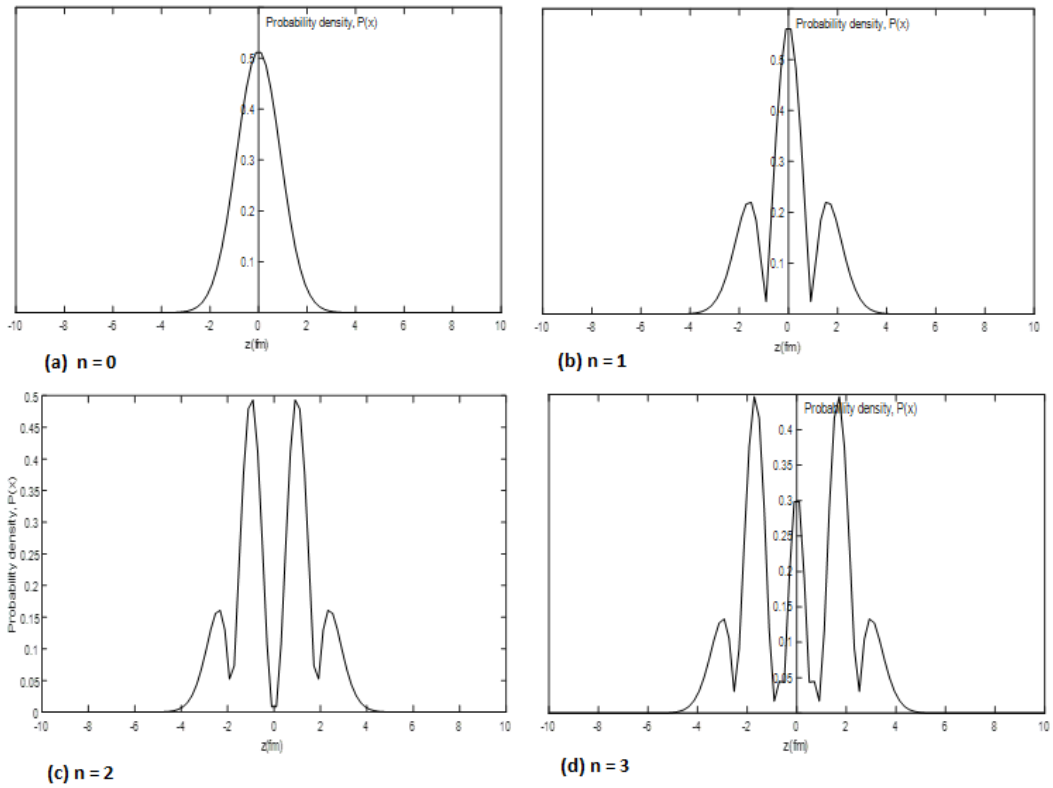

Figure 6. Probability densities of the Radial eigenfunctions of the symmetric modified double-center Wood-Saxon potential for n = 0, 1, 2 & 3.

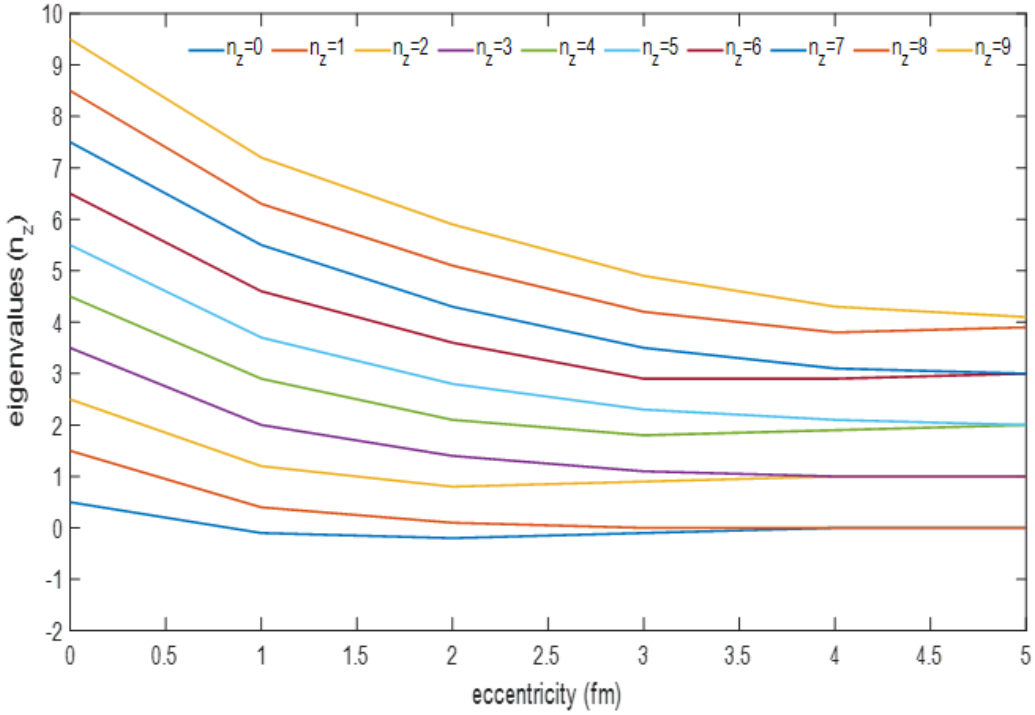

Figure 7. Variation of energy eigenvalues (n_z) against Eccentricity, *zfm*.

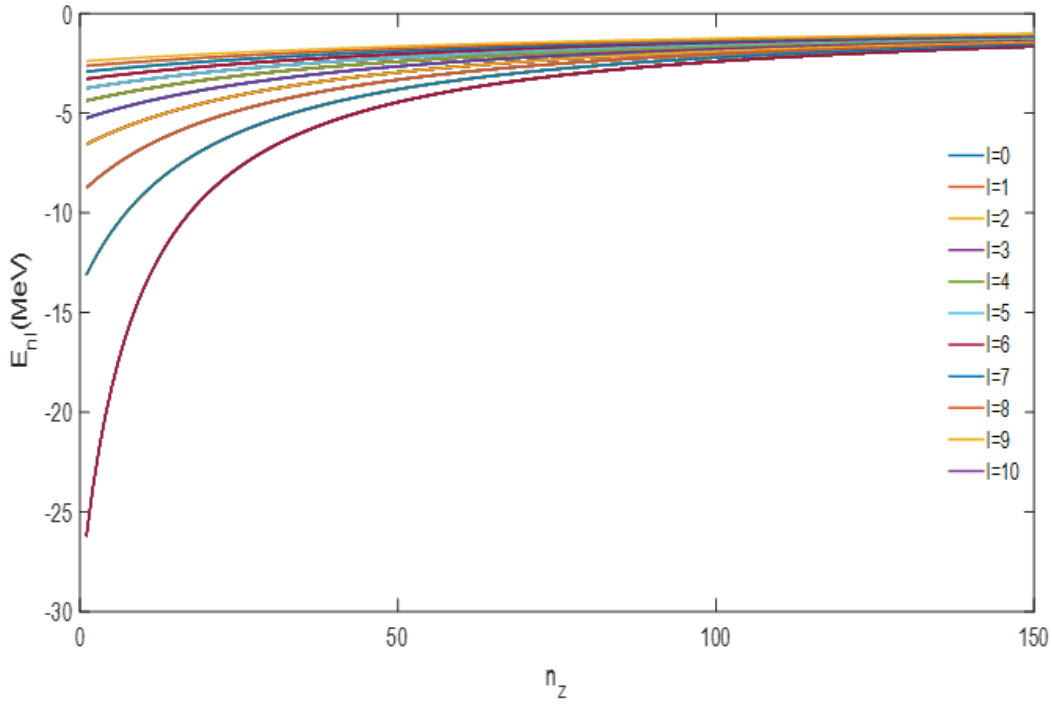

Figure 8. Variation of energy with eigenvalues n_z for different values of the orbital angular momentum l.

The eigenfunction:

$$\psi_{n,l}(\eta) = N_{nl}\,\eta^{c_{12}}(1-c_3\eta)^{c_{13}} \times {}_2F_1\left(-n,\ 1+c_{10}+c_{13}+n;\ c_{10}+1;c_3\eta\right), \tag{6}$$

where ${}_2F_1$ is the Gauss Hypergeometric function and N_{nl} is the normalization constant which is obtained from the normalization

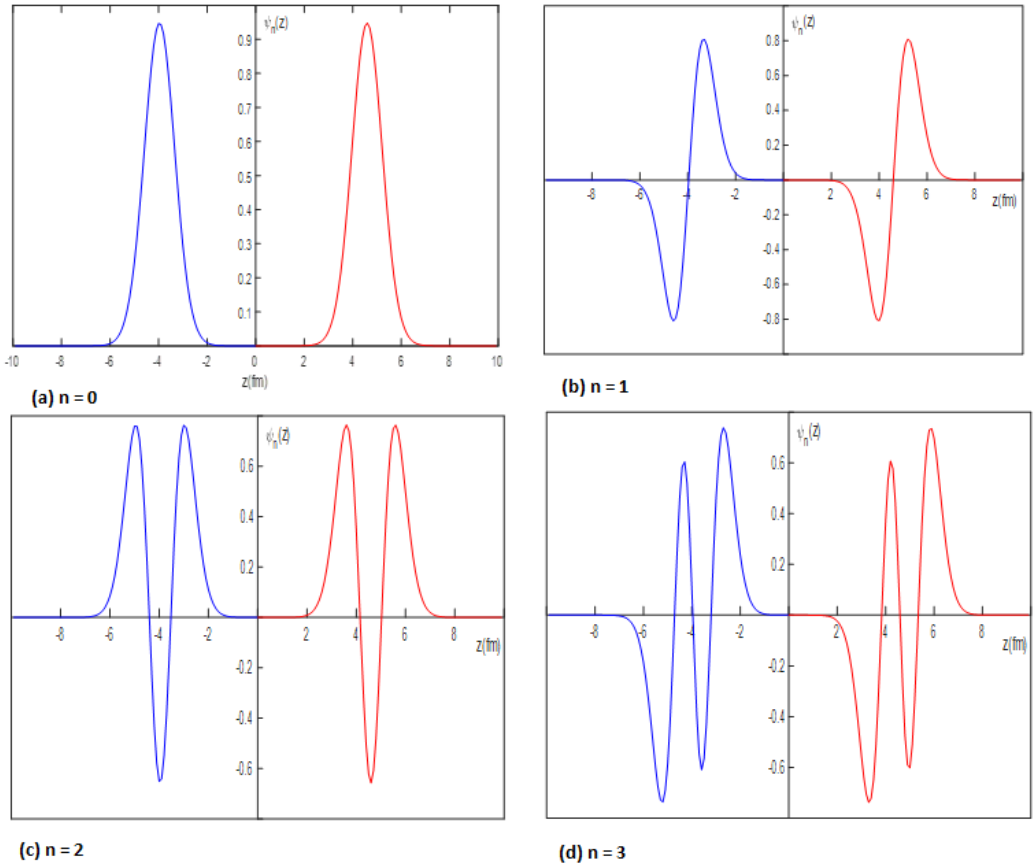

Figure 9. Radial eigenfunctions of the asymmetric modified double-center Wood-Saxon potential for n = 0, 1, 2 & 3.

condition:

$$\int_{-\infty}^{\infty} |\psi_{nl}(r)|^2 dr = 1. \tag{7}$$

The wave function in regions I and II are $\psi_{n,l}(\beta)$ and $\psi_{n,l}(\eta)$ respectively. Simulations were made for the asymmetric splitting of Uranium-235 $^{235}_{92}U$ into two fragments with the starting points being the integers corresponding to a unique center. The eigenfunctions in I and II are assumed to have definite parity i.e. even or odd. The continuity (smooth joining) condition requires that at the boundary (*i.e.* $r = 0$) the slope of the even wave must be equal to zero while the odd wave functions must also vanish there. Thus, matching $\psi_{n,l}(\beta)$ and $\psi_{n,l}(\eta)$ and their derivatives at $x = 0$ thus gives:
For odd parity;

$$\psi_{n,l}(\eta) = -\psi_{n,l}(\beta)\Big|_{r=0},$$

and for even parity;

$$\frac{d\psi_{n,l}(\eta)}{dr} = \frac{d\psi_{n,l}(\beta)}{dr} \quad at\ r{=}0$$

But at $r = 0$,

$$\eta = \beta = e^{-\frac{R_o}{a}} = e^{-\alpha}.$$

Thus our eigenvalue equations for odd parity become:

$$N_{nl}\,(e^{-\alpha})^{c_{12}}(1 - c_3 e^{-\alpha})^{c_{13}} \times {}_2F_1\,(-n,\ 1 + c_{10} + c_{13} + n;\ c_{10} + 1; c_3 e^{-\alpha}) = 0, \tag{8}$$

while for even parity:

$$\begin{aligned}&(e^{-\alpha})^{c_{12}}(1 - C_3 e^{-\alpha})^{c_{13}}\left[\frac{(-n)(1 + c_{10} + c_{11} + n)}{c_{10} + 1}\right]{}_2F_1(-n, 1 + C_{10} + C_{11} + n; C_{10} + 1; C_3 e^{-\alpha})\\ &+ (1 - C_3 e^{-\alpha})^{c_{13}}\,{}_2F_1(-n, 1 + C_{10} + C_{11} + n; C_{10} + 1; C_3 e^{-\alpha}) c_{12} (e^{-\alpha})^{c_{12}+1}\\ &+ (e^{-\alpha})^{c_{12}}\,{}_2F_1(-n, 1 + C_{10} + C_{11} + n; C_{10} + 1; C_3 e^{-\alpha})\,(-c_{13} c_3)\,(1 - C_3 e^{-\alpha})^{c_{13}+1} = 0.\end{aligned} \tag{9}$$

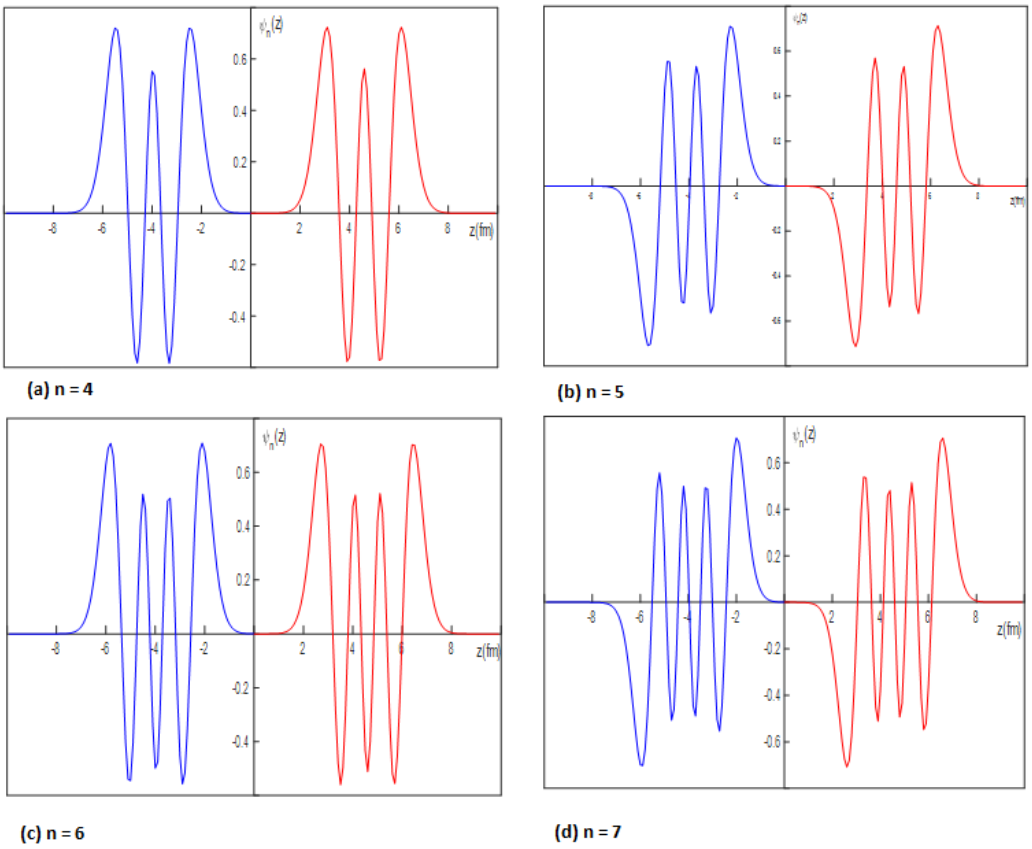

Figure 10. Radial eigenfunctions of the asymmetric modified double-center Wood-Saxon potential for n = 4, 5, 6 & 7.

The transcendental equations (7) and (8) are the eigenvalue equations of the system. The parameter $\alpha = \frac{R_o}{a}$ is the non-dimensional displacement that is varied from 0 to 7 while the energy eigenvalues, n are being computed numerically using root bisection algorithm. It is expected that the energy eigenvalues (n) decrease through non-integer values until at the end of the process, where the quantum numbers emerge as two integer values, specific for each of the totally separated fragment at infinite separation. The eigenfunctions will also be plotted along with their probability densities to reveal the different states of the system and their dynamics.

For the asymmetric case of the fission of $^{235}U^{141}Ba + {}^{94}Kr$, the radii of the two fragments is used as the distance between the centers of the Modified double-centered Wood-Saxon potential and the wave functions, probability densities and energy eigenvalues are again computed. This is illustrated in the Figure 4:

From Figure 4 R_{01} = nuclear radius of Krypton = 3.9623 fm and R_{02} = nuclear radius of Barium = 4.5910 fm. The expression for energy E_{nl} used in this work is that of the lowest energy levels of the Wood-Saxon potential given by Niknam *et al.* [9]:

$$E_{nl} = -\frac{\hbar^2}{2ma^2}\left[\frac{1+2n}{2} - \frac{\left(n(n+1) - \frac{2ma^2 v_o}{\hbar^2 q}\right)}{1+2n+\sqrt{1+\frac{4l(l+1)}{q^2}}}\right]^2 . \; 0 \le n < \infty, \; q \ge 1 \tag{10}$$

3. Result and discussion

3.1. Normalized radial eigenfunctions and probability densities of the symmetric modified double center Wood-Saxon potential

The normalized radial eigenfunctions $R_{nl}(z)$ of the modified double center Wood-Saxon potential for the $l = 0$ state are presented in Figures 5 and 6. It is worthy to note that number of nodes is equal to the principal quantum number, n_z in each of the regions (I & II). The probability densities for states n = 0, 1, 2, 3 & 4 are presented in Figure 7.

The results in Figures 5 & 6 showed the normalized radial eigenfunctions of the symmetric Modified double-center Wood-Saxon potential. It was observed that the eigenfunctions exhibit even parity π^+ for eigen values; $n_z = 0, 2, 4, 6$ and odd parity π^- for $n_z = 1, 3, 5, 7$ Also, it can be seen that the number of nodes characterizes the state of the eigenfunctions [9]. The probability densities P(r), plotted in Figure 7 shows the likelihood of locating a particle of energy, E, either in Region I or II of the Modified

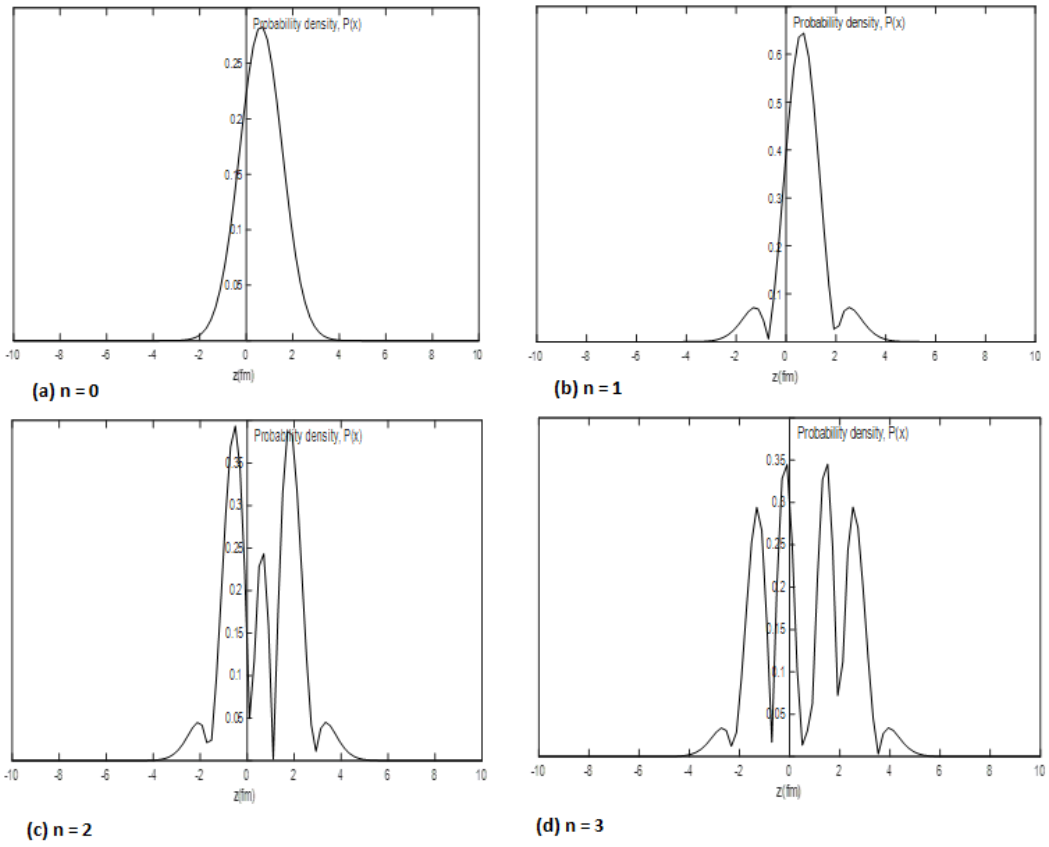

Figure 11. Probability densities of the Radial eigenfunctions of the asymmetric modified double-center Wood-Saxon potential for n = 0, 1, 2 & 3.

double-centered Wood-Saxon potential well. As a result of the symmetry of the system and the reduced mass, it is observed that there is equal probability of the particle being located in any of the regions. The highest value of probability is at the center (r = 0) which is the equilibrium or rest position of the particle and at distances away from the center, the probability attenuates. This because the particle requires more energy to separate from the parent nuclei (uranium) and become a new completely separated daughter nuclei.

3.2. Energy eigenvalues of the symmetric modified double center Wood-Saxon potential

The plot in Figure 8 shows the variation of energy eigenvalues with eccentricity, z (fm) for the symmetric double-center Wood-Saxon potential. The result from the plot shows that the transition from integer values at z = 0, convergence through separate schemes from z = 1 to 3 and return to integer values beyond z = 4. The two pairs of spectral lines which converge to integer values of lower eigenvalues indicates double fold degeneracy in the symmetric Modified double center Wood-Saxon potential.

The results in Figure 9 shows the variation of energy with eigenvalues n_z for different values of the orbital angular momentum, l. It can be observed that the energy diminishes with and tends to a constant value for higher values of the principal quantum number n_z. This is reflective of the conservation of energy and angular momentum in the system.

The results in Figures 8 shows a plot of the energy eigenvalues, n_z against the eccentricity, $z = \frac{R_0}{a_0}$. For the symmetric case in Figure 8, at z = 0, the fission fragments are not yet formed as the position marks the onset of the fission process, thus the system is still in equilibrium and so they overlap. At this stage, the vibrational frequency $\omega \approx 0$ and thus the system behaves like a single mass (the reduced mass). As a result of this, the energy eigenvalues are half integral for both even and odd parities. This is similar to the case of the double-centered oscillator. As the eccentricity increases, the vibrational frequency increases. Two separate lines now appear in the energy spectrum from across different levels which diverge initially but then begin to converge as $z \rightarrow R_o$, which is the distance from the center. At this stage, the energy eigenvalues, n_z become non-integral, but become half integral again as z > R_o, while at eccentricities where z >> R_o, the eigenvalues become half integral again, an indication that the two nuclei are now completely separated and both exist in independent potential wells of their own.

3.3. Normalized radial eigenfunctions and probability densities of the asymmetric modified double center Wood-Saxon potential

The asymmetric case of nuclear fission was simulated using the splitting of ^{235}U to give ^{141}Ba and ^{84}Kr. The radii of the two fragments were set as follows: R_{01} = 3.9623 fm (Krypton) to the left and R_{02} = 4.5910 fm (Barium) to the right. These values

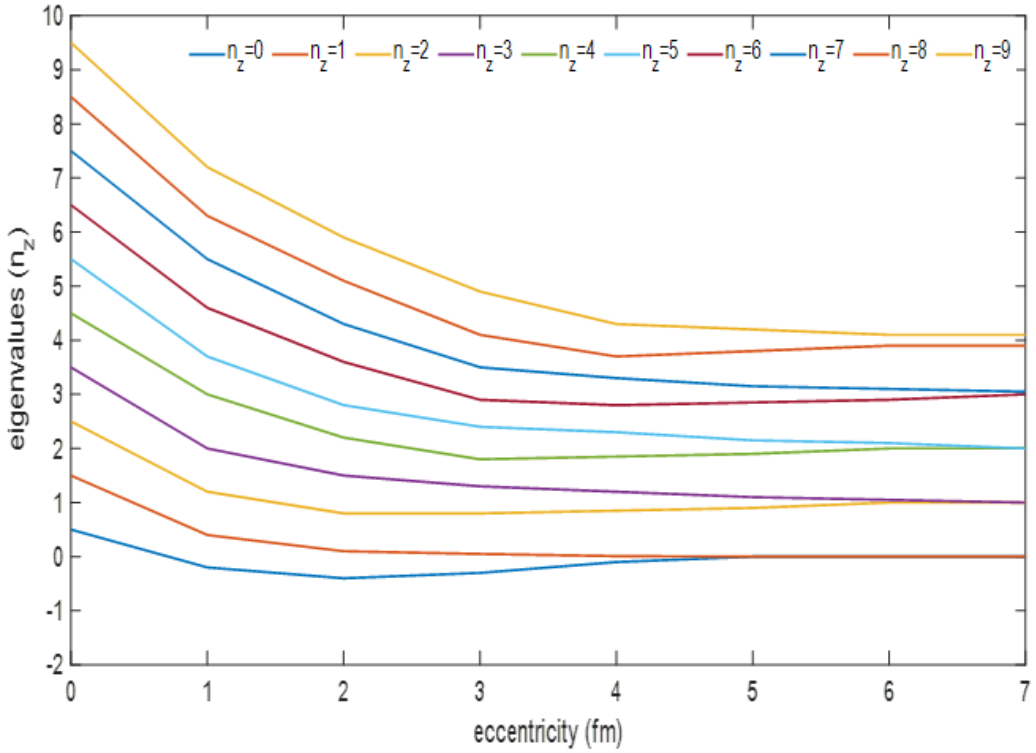

Figure 12. Variation of energy eigenvalues (n_z) against eccentricity, z (Fm).

were used as the distance the centers of the double-centered Wood-Saxon potential in regions I and II. It can be observed that the eigenfunctions look identical to those in Figure as the difference in the distance from the centers in regions I and II is very small (0.6287 fm). Thus the case of ^{235}U fission can be considered a pseudo-symmetric case of the double centered Wood-Saxon potential. The radial eigenfunctions $R_{nl}(z)$ of the modified double center Wood-Saxon potential for the $l = 0$ state are presented in Figures 10 and 11 while the probability densities $\left(P(z) = |R_{nl}(z)|^2\right)$ are shown in Figure 12. The shift in the probability densities more to the right is as a result of the asymmetric separation in the centers of the two potentials with the ^{141}Ba nuclei to the right having a larger nuclear radius.

The results in Figures 10 & 11 show the normalized radial eigenfunctions of the asymmetric Modified double-center Wood-Saxon potential for the nuclear fission of ^{235}U. It is also observed that the eigenfunctions also exhibit even parity π^+ for $n_z = 0,\ 2,\ 4, 6\ldots$ and odd parity π^- for $n_z = 1,\ 3,\ 5,\ 7\ldots$ with a slight bias to the right. This is not visible in the eigenfunctions plots as the nuclear mass difference between the fragments (Ba and Kr) is quite small (but not negligible). However, the slight shift to the right (towards the Ba nuclei) is more conspicuous in the plot of probability densities in Figure 12. Also, it can be seen that the number of nodes characterizes the state of the eigenfunctions. Furthermore, the probability densities P(r), plotted in Figure 12 shows that as a result of the asymmetry in the size and mass of the fragments, the probability of locating the reduced mass to the right of the potential well (Region I) is higher than the left side (Region I). The peak value of probability is shifted slightly beyond the equilibrium or rest position in to the right-hand side in region II and beyond that, the probability attenuates. Thus, the separation of fragments is more skewed towards the Barium nucleus.

3.4. Energy eigenvalues of the asymmetric modified double center Wood-Saxon potential

The plot in Figure 12 shows the variation of energy eigenvalues with eccentricity, z (fm) for the asymmetric double-center Wood-Saxon potential applied to binary fission of ^{235}U. The result from the plot shows similar trends as observed in Figure 8 as this case is a pseudo-symmetric system.

The results in Figures 12 shows a plot of the energy eigenvalues, n_z against the eccentricity, $z = \frac{R_0}{a_0}$.

Furthermore, from the results in Figure 12, the asymmetric splitting of ^{235}U into two fragments; ^{141}Ba and ^{94}Kr is simulated using the Modified double-center Wood-Saxon potential. Here too, at z = 0, the fission fragments are not yet formed at the onset of the fission process, thus the system is still in equilibrium. The vibrational frequency $\omega \approx 0$ and thus the system behaves like a single mass (the reduced mass). As a result of this, the energy eigenvalues are half integral for both even and odd parities. As the Uranium nucleus is bombarded by the neutron, it gets excited after fusion to form the highly unstable ^{236}U nucleus which the vibrates and begins to disintegrate into the two fragments plus a neutron. Thus, as the splitting begins, eccentricity increases and the

Table 1. Summary of findings.

Model	Strengths	Weaknesses
Standard Woods-Saxon Potential	Realistic nuclear mean-field, finite range	Fails in fission scenarios, does not separate into two wells
Two-Center Shell Model	Useful in fusion and fission, allows separate wells	Uses harmonic oscillator potential, lacks finite-range properties
Double-Center Oscillator Model	Early fission model, two potential centers	Potential extends to infinity, unrealistic surface effects
Niknam *et al*. Numerical Approach	Solves Woods-Saxon potential using Nikiforov-Uvarov method	Does not account for fission fragment separation
MDCWSP (Present work)	Accurately models fission, captures degeneracy, smooth energy Transition	Requires computational complexity for numerical solutions

vibrational frequency increases. Two separate lines now appear in the eigenvalue spectrum from across different states which diverge initially through non-integer values of n_z but then begin to converge as $z \rightarrow R_o$, which is the distance from the center and also the radius of the Uranium nucleus (Ro = 5.41722 fm). The convergence of the energy levels is a proof of double fold degeneracy in the pseudo-symmetric double centered asymmetric Wood-Saxon potential. At this stage, the energy eigenvalues, n_z become non-integral and tend towards half integral values as $z > R_o$, while at eccentricities where $z >> R_o$, the eigenvalues become half integral again, an indication that the two nuclei of Ba and Kr are now fully separated and both exist in completely different potential wells of their own. Thus, double fold degeneracy is also involved. The difference between the symmetric and asymmetric Modified double-center Wood-Saxon potential is the rate of convergence of the spectral lines as $z \rightarrow R_o$. As the higher degrees of asymmetry, the lines do not merge together but tends towards integer or half integer values separately. Thus, no degeneracy is observed in such case [4].

Finally, the total radial energy plotted in Figure 9 was computed for $l = 0 - 10$. Convergence of energy is observed as n_z increases for all values of orbital angular momentum, l. This is another confirmation of the degeneracy of the system from the angular coordinates as the different states with different angular momenta have the same energy on the long run.

3.5. Comparative analysis with other theoretical models

To assess the validity and improvements introduced by the Modified Double-Center Woods-Saxon Potential (MDCWSP), we compare its results with several established theoretical models, including the conventional Woods-Saxon potential, the Two-Center Shell Model, the Double-Center Oscillator Model, and previous numerical approaches using the Nikiforov-Uvarov method.

The traditional Woods-Saxon potential is widely used to describe the mean-field interaction in nuclei. However, it assumes a single well-defined potential, which becomes less effective in modeling binary fission, where two distinct nuclear fragments are formed. Our modified double-center approach introduces. A dual-potential structure that better represents the transition from a single nucleus to two separate fission fragments. More accurate eigenvalue distributions that capture the energy degeneracy observed in fission processes. A potential landscape that does not diverge at high deformations, unlike the single-center Woods-Saxon model.

The Two-Center Shell Model (TCSM) is used primarily to describe fusion and fission processes by considering the interaction between two potential wells [2]. The key differences between MDCWSP and TCSM are: TCSM relies on a harmonic oscillator potential, which lacks the realistic finite-range properties of the Woods-Saxon potential. MDCWSP incorporates a smooth transition between the fissioning nucleus and separated fragments, providing a more physically accurate description of nuclear interactions. The eigenfunctions obtained from MDCWSP show double-fold degeneracy, which is not explicitly addressed in the TCSM.

The Double-Center Oscillator Model (DCOM) assumes two harmonic oscillators representing the nuclear fragments [4]. While useful in early fission studies, it presents limitations: The harmonic oscillator potential extends to infinity, making it less suitable for finite nuclear interactions. MDCWSP, with its Woods-Saxon form, provides a more realistic description of nuclear surface effects and short-range interactions.

The energy eigenvalues in MDCWSP vary smoothly with eccentricity, capturing fission dynamics more accurately than the stepwise transitions in DCOM.

From these comparisons, the MDCWSP proves to be a more effective model for describing the binary fission of heavy nuclei such as uranium-235. It successfully incorporates key nuclear properties, including short-range interactions, finite-range effects, and degeneracy in energy levels, making it a valuable tool for nuclear fission studies.

4. Conclusion

A modified double-center Wood-Saxon potential was modeled and substituted into the radial part of the three-dimensional Schrödinger equation. The Schrödinger equation was then solved using the Nikiforov-Uvarov method with the reduced mass of

the fission products (^{141}Ba and ^{84}Kr) and radius of Uranium nucleus inputted to obtain the eigenfunctions of the double-center Wood-Saxon potential and their probability densities for the fission of ^{235}U. It was observed that, the eigenfunctions exhibited even parity π^+ for eigenvalues; $n_z = 0,\ 2,\ 4, 6\ldots$ and odd parity π^- for $n_z = 1,\ 3,\ 5,\ 7\ldots$ with the number of nodes characterizing the state of the eigenfunctions. The energy eigenvalues were obtained by applying the continuity conditions at the boundary (z = 0) and solving the resultant transcendental equations at different values of eccentricity. The spectrum of eigenvalues plotted showed double fold degeneracy as the transition from two independent schemes into one at infinite separation describes the complete asymmetric fission of ^{235}U into two fragments.

Data Availability

In accordance with the university's policies, the data is available through the school.

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