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An Inertial Algorithm of Generalized f - Projection for Maximal Monotone Operators and Generalized Mixed Equilibrium Problems in Banach Spaces

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Abstract

In this paper, we study a modified hybrid inertial algorithm of generalized f- projection for approximating maximal monotone operators and solutions of generalized mixed equilibrium problems in Banach spaces. Our results generalize and improve many recent announced results in the literature.

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1. Introduction

An inertial algorithms is a method of speeding up the convergence of the sequence of an iterative algorithm which was first introduced and studied by Polyak [1]. An inertial-types algorithm is a two-step iterative techniques in which the next iteration is defined by making use of the previous two iterates. Consequently, many researches involving inertial-type algorithm are now taking place (see, e.g [2, 3, 4, 5] and the references therein).

Let C be a nonempty closed convex subset of a real Banach space E with $||.||$ and E^* as the norm and dual space of E respectively. Let $\Psi : C \times C \longrightarrow \mathbb{R}$ be a bifunctions, where $\mathbb{R}$ is the set of real numbers, $\Phi : C \longrightarrow E^*$ is a nonlinear

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continuous monotone mapping and $\varphi : C \longrightarrow \mathbb{R}$ be a convex and lower semi continuous function. The generalized mixed equilibrium problem [6] is to find $x \in C$ such that:

$$\Psi(x,y) + \langle \Phi x, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C.$$

The set of solutions of generalized mixed equilibrium problem is denoted by

$$GMEP(\Psi,\Phi,\varphi) = \{x \in C : \Psi(x,y) + \langle \Phi x, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C\}.$$

The generalised mixed equilibrium problem has been used as a tools and unified approach for investigating and solving a large number of problems arising from nonlinear analysis, optimization, economics, mathematical physics, game theory and variational inequality problem and so forth (see [7, 8] and the references therein). Recently, the equilibrium problem has been extensively investigated based on hybrid algorithms, in particular, the monotone hybrid algorithm; see [9, 10, 11] and the references therein.

Let S be a maximal monotone operator from E to E^*. The problem of a zero point of a maximal monotone operator is to find a point $\omega \in E$ such that

$$0 \in S(\omega). \tag{1}$$

We denote $S^{-1}0$ as the set of all point $\omega \in E$ such that $0 \in S(\omega)$. This problem play an important role in analysis, optimization and other related field of research.

Martinet [12] was the first to introduced the proximal point algorithm (PPA) which is well known as the classical techniques for approximating (1). With regards to this important, a number of researches have been working on (PPA) techniques (see for example [12, 13, 14] and the references therein). Solodov and Sviater [13] studied a modified proximal point algorithm and projection in Hilbert space. In 2003, Kohsake and Takahashi [15] proposed and established strong convergence results for maximal monotone operators in Banach space. Alber [16, 17] proposed and proved the generalized projections $\Pi_C : E^* \longrightarrow C$ and $\Pi_E : E \longrightarrow C$ in uniformly smooth and uniformly convex Banach space. In 2005 Li [18] proved strong convergence theorem for generalized projection in a reflexive Banach space. In 2010 Li *et al.* [19] studied the generalized f- projection operator and established strong convergence results for relatively nonexpansive mappings in Banach spaces.

In 2012, Siwaporn and Kumam [20] considered the following hybrid iterative algorithm of generalized f- projection operator for approximating the set of two countable families of weak relatively nonexpansive mappings and the set of solutions of generalized Kly Fan inequalities in a uniformly smooth and uniformly convex Banach space:

$$\begin{cases} C_1 = C, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ v_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS_n u_n), \\ z_n = S^{\Gamma_m}_{r_{m,n}} S^{\Gamma_{m-1}}_{r_{m-1,n}} \dots S^{\Gamma_2}_{r_{2,n}} S^{\Gamma_1}_{r_{1,n}} u_n, \\ C_{n+1} = \{z \in C : G(z, Jz_n) \leq G(z, Jv_n) \leq G(z, Ju_n) \leq G(z, Jx_n) \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \forall n \geq 1. \end{cases}$$

They proved that the sequence $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = \Pi^f_{\Omega} x_0$
Chidume *et al.* [21] proved a strong convergence theorem for generalized ϕ- strongly monotone maps in uniformly convex and uniformly smooth Banach spaces. Also, in 2020, Chidume *et al.* [5] studied the following hybrid inertial algorithm for approximating a point in the set of zero of a maximal monotone and a common fixed point of a countable family of relatively nonexpansive mapping in Banach spaces:

$$\begin{cases} C_0 = E, \\ w^n = v^n + \gamma_n(v^n - v^{n-1}), \\ u^n = J^{-1}((1-\beta)Jw^n + \beta T J_{r_n} w^n), \\ z^n = J^{-1}((1-\alpha)Jw^n + \alpha S u^n)), \\ C_{n+1} = \{z \in C : \phi(z, z^n) \leq \phi(z, Jw^n), \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \forall n \geq 0. \end{cases}$$

They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_\Gamma x_0$.
Recently, Hammad *et al.* [22] studied a hybrid algorithm for approximating zero of the sum of maximal monotone operators and common fixed point problem for finite family of relatively quasi- nonexpansive mappings in Banach space.
Very recently, Siwaporn Soewan [23] introduced and studied the following new hybrid iterative algorithm for approximating maximal monotone operators by considering the notion of generalized f- projection in Banach spaces:

$$\begin{cases} x_1 \in C,\ C_1 = C, \\ z_n = J^{-1}(\gamma_n Jx_n + (1-\gamma_n)JJ_{r_n}x_n), \\ C_{n+1} = \{z \in C : G(z, Jz_n) \leq G(z, Jx_n), \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1, \forall n \geq 1. \end{cases}$$

The author proved that the sequence $\{x_n\}$ converges strongly to $\Pi^f_{T^{-1}0} x_1$.

Motivated inspired by the results of Chidume *et al.* [5], Siwaporn and Kumam [20], and Siwaporn Saewan [23] mentioned above, we study a modified hybrid inertial algorithm of generalized f- for approximating a zero point of a maximal monotone operators and solutions of generalized mixed equilibrium problems in Banach space. Our results extends and improves the result of Siwaporn Saewan [23] and many results in the literature.

2. Preliminaries

Let E be a real Banach space with norm $\|\,.\,\|$, E^* be the dual space of E, let C be a nonempty closed convex subset of E. The normalized duality mapping on E is a mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E,$$

where $\langle x, x^* \rangle$ is the pairing between element of E and that of E^*.
Let $D := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space s E is said to be smooth if the $\lim_{t\to 0} \frac{\| x + ty \| - \| x \|}{t}$ exists for all $x, y \in D$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in D$, E said to be strictly convex if $\frac{\| x + y \|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$ and E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\| x + y \|}{2} \leq 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.
The modulus of convexity of E is the function $\delta : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = inf\{1 - \|\frac{x + y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}$$

Let E be a smooth Banach space. Define a map $\phi : E \times E \longrightarrow \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \forall x, y \in E. \tag{2}$$

It follows from (2), that

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E \tag{3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad \forall x, y, z \in E \tag{4}$$

and

$$\phi(x, y) \leq \|x\|\|Jx - Jy\| + \|y\|\|x - y\|, \quad \forall x, y, \in E \tag{5}$$

Following Alber [16, 24], the generalised projection Π_C from E onto C is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C(x) = x^*$, where x^* is the solution to the minimization problem

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x).$$

Existence and the uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. If E is a real Hilbert space H, then $\phi(y, x) = \|y - x\|^2$ and Π_C become the metric projection P_C of H onto C (see, for example [5, 25, 26]).

Remark 2.1. *Let E be a Banach space. We recall from the following [23] that:*

i. If E is an arbitrary Banach space, then J is monotone and bounded;
ii. If E is a smooth, then J is single valued and semi continuous;
iii. If E is a strictly convex, then J is strictly monotone;
iv. If E is reflexive, smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto;
v. If E is uniformly smooth, then E is smooth and reflexive;
vi. E is uniformly smooth if and only if E^* is uniformly convex;
vii. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subset of E.

Remark 2.2. *If E is a reflexive, smooth and strictly convex Banach space, the for $x, y \in E$, we have that $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that for $\phi(x, y) = 0$, we get that $x = y$. It follows from (ii) above that $\| x \|^2 = \| y \|^2$. Which implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. Now by the definition of J, we conclude that $Jx = Jy$. Thus, this implies that $x = y$ (see for example [23, 27] and therein)*

Definition 2.3. *i) Let E be a strictly convex, smooth and reflexive Banach space, let S be a set valued from E to E^* denoted by $S \subset E \times E^*$ and the graph $G(S) = \{(x, y) : y \in Sx\}$. We denote $D(S) = \{x \in E : Sx \neq \emptyset\}$ and $R(S) = \cup\{Sx : x \in D(S)\}$ as the Domain and Range of an operator S respectively.*

ii) An operator $S \subset E \times E^*$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in S$.

iii) A monotone operator S is said to be maximal if its graph $G(S)$ is not properly contained in the graph of any other monotone operator. Recall that if S is a maximal monotone operator, then it follows that $S^{-1}0 = \{x \in D(S) : 0 \in Sx\}$ which is closed and convex. It is well known that S is a maximal monotone if and only if $R(J + rS) = E^*$ for all $r > 0$. The resolvent of S is denoted by $J_r = (J + rS)^{-1}J$ for all $r > 0$, where J_r is a single valued mapping from E to $D(S)$. Furthermore $S^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ denote the set of all fixed point of J_r. The Yosida approximation of S is defined by $S_r = (J - JJ_r)/r$, for all $r > 0$. We can also recall that $S_r x \in S(J_r x)$ for all $r > 0$ and $x \in E$.

iv) An operator S is said to be closed if for any sequence $\{x_n\} \subset C$, with $x_n \longrightarrow x$ and $Sx_n \longrightarrow y$ then $y = Sx$.

For solving the generalized mixed equilibrium problem GMEP(Ψ, Φ, φ)[28, 29], we assume that the nonlinear mapping $\Phi : C \longrightarrow E^*$ is continuous and monotone, the function $\varphi : C \longrightarrow \mathbb{R}$ is convex and lower semi-continuous and the bifunction, $\Psi : C \times C \longrightarrow \mathbb{R}$ satisfies the following conditions:
(L_1) $\Psi(x, x) = 0, \ \forall x \in C$;
(L_2) $\Psi(y, x) + \Psi(x, y) \leq 0 \ \forall x, y \in C$;
(L_3) $\Psi(x, y) \geq \limsup_{\lambda \downarrow 0} \Psi(\lambda z + (1 - \lambda)x, y), \forall x, y, z \in C$;
(L_4) $y \mapsto \Psi(x, y)$ is convex and weakly lower semi-continuous, $\forall x \in C$.

The following lemmmas play important roles in this paper.

Lemma 2.4. *(see [30]) Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \to \infty} \| x_n - y_n \| = 0$.*

Remark 2.5. *If $\{x_n\}$ and $\{y_n\}$ are bounded, then by considering (5) it is obvious that the converse of Lemma 2.4 is also true.*

Let $G : C \times E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a functional defined by:

$$G(y,\rho) = \| y \|^2 - 2\langle y,\rho\rangle + \|\rho\|^2 + 2\sigma f(y), \tag{6}$$

where $y \in C, \rho \in E^*, \sigma$ is positive number and $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi continuous. It follows from the definitions of G and f that the following properties hold:
i) $G(y,\rho)$ is convex and continuous with respect to ρ when y is fixed;
ii) $G(y,\rho)$ is convex and lower semicontinuous with respect to y when ρ is fixed.

Let C be a nonempty closed convex subset of a real Banach space E. The generalized f- projection $\Pi_C^f : E^* \longrightarrow 2^C$ is an operator defined by

$$\Pi_C^f \rho = \{v \in C : G(v,\rho) = \inf_{y\in C} G(y,\rho), \forall \rho \in E^*\}.$$

Lemma 2.6. *(see [31]) Let E be a reflexive Banach space with its dual E^* and C be a nonempty closed convex subset of E. The following statements hold:*
i) $\Pi_c^f \rho$ is nonempty closed convex subset of C for all $\rho \in E^*$;
ii) If E is smooth, then for all $\rho \in E^*, x \in \Pi_C^f \rho$ if and only if

$$\langle x - y, \rho - Jx\rangle + \sigma f(y) - \sigma f(x) \geq 0, \forall y \in C;$$

iii) If E is strictly convex and $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$ such that $\lambda x \in C$ where $x \in C$), then Π_C^f is single valued mapping.

Lemma 2.7. *(see [32]) Let C be nonempty closed convex subset of a reflexive Banach space E and E^* be the dual space of E. If E is strictly convex, then $\Pi_C^f \rho$ is single valued.*

It is well known that if E is a smooth Banach space, then J is single valued mapping. Therefore, there exists a unique element $\rho \in E^*$ such that $\rho = Jx$ for $x \in E$. Now, by substituting $\rho = Jx$ in (6), we obtain

$$G(y, Jx) = \| y \|^2 - 2\langle y, Jx\rangle + \|x\|^2 + 2\sigma f(y). \tag{7}$$

It follows from the definition of G that

$$G(y, Jx) = G(y, Jz) + \phi(z, x) + 2\langle y - z, Jz - Jx\rangle, \forall x, y, z \in E. \tag{8}$$

Furthermore, we consider the notion of the second generalized f- projection in Banach spaces,

Definition 2.8. *(see [19]) Let C be a nonempty closed convex subset of a real smooth Banach space E. Then, an operator $\Pi_C^f : E \longrightarrow 2^C$ is said to be generalized f- projection if*

$$\Pi_C^f x = \{v \in C : G(v, Jx) = \inf_{y\in C} G(y, Jx), \forall x \in E\}.$$

Lemma 2.9. *(see [33]) Let E be a Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicountinuous convex functional. Then there exists $q^* \in E^*$ and $\gamma \in \mathbb{R}$ such*

$$f(x) \geq \langle x, q^*\rangle + \gamma, \forall x \in E.$$

Lemma 2.10. *(see [19]) Let C be a nonempty closed convex subset of a reflexive smooth Banach space E. Then, the following statements hold:*
i) $\Pi_C^f x$ is nonempty closed convex subset of C for all $x \in E$;
ii) for all $x \in E$, $\hat{x} \in \Pi_C^f$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \sigma f(y) - \sigma f(\hat{x}) \geq 0, \forall y \in C;$$

iii) If E is strictly convex, then Π_C^f is single valued mapping.

Lemma 2.11. *(see [19]) Let C be a nonempty closed convex subset of a reflexive smooth Banach space E. and $\hat{x} \in \Pi_C^f$ for all $x \in E$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \forall y \in C.$$

Lemma 2.12. *(see [19]) Let E be a Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-countinuous mapping with domain $D(f)$. If $\{x_n\} \subset D(f)$ such that $x_n \rightharpoonup \hat{x} \in D(f)$ and $G(x_n, Jy) \longrightarrow G(\hat{x}, Jy)$ (as $n \to \infty$), then $\| x_n \| \longrightarrow \| \hat{x} \|$ (as $n \to \infty$).*

Lemma 2.13. *(see [15]) Let C be a nonempty closed convex subset of strictly convex, smooth and reflexive Banach space E, let $S \subset E \times E^*$ be a monotone operator satisfying $D(S) \subset C \subset J^{-1}(\cap_{r>0} R(J + rS))$. Let J_r and S_r, for all $r > 0$ be the resolvent and the Yosida approximation of S, respectively. The following statements hold:*
i) $\phi(v, J_r x) + \phi(J_r x, x) \leq \phi(v, x), \forall x \in C, v \in S^{-1}0$;
ii) $(J_r x, S_r x) \in S, \forall x \in C$, where $(x, x^*) \in S$ denotes the value of x^* at $x(x^* \in Sx)$ iii) $F(J_r) = S^{-1}0$.

Lemma 2.14. *(see [23]) Let E be a strictly convex, smooth and reflexive Banach space, $S \subset E \times E^*$ be a monotone operator with $S^{-1}0 \neq \emptyset$, and for each $r > 0$, $J_r = (J + rS)^{-1}J$. Then*

$$G(q, JJ_r x) + \phi(J_r x, x) \leq G(q, Jx), \forall x \in E, q \in S^{-1}0.$$

Lemma 2.15. *(see [7, 26]) Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $\Psi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions $(L_1) - (L_4)$. Let $r > 0$ be any given number and $x \in E$ be any given point. Then, there exists $z \in C$ such that*

$$\Psi(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C.$$

Replacing x with $J^{-1}(Jx - r\Phi x)$, where B is a monotone mapping from C into E^*, then there exists $z \in C$ such that

$$\Psi(z, y) + \langle y - z, \Phi z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0.\forall y \in C.$$

Lemma 2.16. *(see [26, 29, 34]) Let E be a uniformly smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $\Phi : C \longrightarrow E^*$ be a continuous and monotone mapping, $\Psi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions $(L_1) - (L_4)$ and $\varphi : C \longrightarrow \mathbb{R}$ be a proper convex and lower semi-continuous function. Let $r > 0$ be any given number and $x \in E$ be any given point, define a mapping $T_r : E \longrightarrow C$ by*

$$T_r(x) = \{z \in C : \Psi(z, y) + \varphi(y) - \varphi(z) + \langle y - z, \Phi z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}, \forall x \in E,$$

for all $x \in C$. The mapping T_r has the following properties:
(a) T_r is single-valued;
(b) T_r is a firmly nonexpansive - type mapping, for all $x \in E, y \in C$

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - JTy\rangle$$

(c) $F(T_r) = GMEP(\Psi, \Phi, \varphi)$;
(d) $GMEP(\Psi, \Phi, \varphi)$ is a closed convex set of C.
(e) $\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x), \quad \forall p \in F(T_r), \quad x \in E$.

3. Strong Convergence Theorem

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex real Banach space and C be a nonempty closed and convex subset of E. Let $\Psi : C \times C \longrightarrow \mathbb{R}$ be a bi function which satisfies conditions $(L_1) - (L_4)$, $\Phi : C \longrightarrow E^*$ be continuous and monotone, and $\varphi : C \longrightarrow \mathbb{R}$ be a lower semi-continuous and convex function. Let $f : E \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset int(D(f))$, where $D(f)$ is the domain of f. Let $S \subset E \times E^*$ be a maximal monotone operator satisfying $D(S) \subset C$ and $J_{r_n} = (J + r_n S)^{-1} J$, for all $r_n > 0$. Assume that $\Omega := GMEP(\Psi, \Phi, \varphi) \cap S^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0 = E; \\ w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ u_n = J^{-1}(\beta_n J w_n + (1-\beta_n) J J_{r_n} w_n); \\ z_n \in C \text{ such that } \Psi(z_n, y) + \langle \Phi z_n, y - z_n \rangle + \varphi(y) - \varphi(z_n) \\ +\dfrac{1}{r_n}\langle y - z_n, J z_n - J u_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : G(z, Jz_n) \leq G(z, Jw_n)\}; \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ \forall n \in \mathbb{R} \cup \{0\} \end{cases} \tag{9}$$

where $\alpha_n \subset (0,1)$, β_n is a sequence in $[0,1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\liminf\limits_{n\to\infty}(1-\beta_n) > 0$ and $\lim\limits_{n\to\infty} r_n = \infty$. The sequence $\{x_n\}$ converges strongly to $\Pi^f_\Omega x_0$, where Π^f_Ω is the generalized f- projection of E onto Ω.

Proof.
Let two functions $\Gamma : C \times C \longrightarrow \mathbb{R}$ and $T_r : E \longrightarrow C$ be defined by

$$\Gamma(x,y) = \Psi(x,y) + \langle \Phi x, y - x \rangle + \varphi(y) - \varphi(x), \ \forall x, y \in C$$

and

$$T_r(x) = \{u \in C : \Gamma(u, y) + \frac{1}{r_n}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C\} \ \forall x \in E,$$

respectively. Then, the function Γ satisfies conditions $(L1) - (L4)$ and T_r has the properties $(a) - (e)$ of Lemma 2.16 (see [26, 34, 29]). Therefore iterative sequence (9) can be rewritten as

$$\begin{cases} x_0 \in C_0 = E; \\ w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ u_n = J^{-1}(\beta_n J w_n + (1-\beta_n) J J_{r_n} w_n); \\ z_n \in C \text{ such that } \Gamma(z_n, y) + \dfrac{1}{r_n}\langle y - z_n, J z_n - J u_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : G(z, Jz_n) \leq G(z, Jw_n)\}; \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ \forall n \in \mathbb{R} \cup \{0\} \end{cases} \tag{10}$$

We first show that $\Omega \subset C_n, \forall n \geq 0$ and $\{x_n\}$ is well defined. Assume that C_n is closed and convex for all $n \in \mathbb{N}$. Now by the definition of C_{n+1}, for any $z \in C_n$, we have

$$G(z, Jz_n) - G(z, Jw_n) \leq 0.$$

which gives that

$$\| z \|^2 - 2\langle z, Jz_n \rangle + \|z_n\|^2 + 2\sigma f(z) - \| z \|^2 + 2\langle z, Jw_n \rangle - \|w_n\|^2 - 2\sigma f(z) \leq 0.$$

This implies that

$$2\langle z, Jw_n \rangle - 2\langle z, Jz_n \rangle + \|z_n\|^2 - \|w_n\|^2 \leq 0,$$

thus

$$2\langle z, Jw_n - Jz_n\rangle \leq \| w_n \|^2 - \| z_n \|^2 .$$

Hence, C_{n+1} is closed and convex, $\forall n \geq 0$. Therefore, $\Pi^f_{C_{n+1}} x_0$. is well defined.

Next, we now show that $\Omega \subset C_n$. Assume that $z_n = T_{r_n} u_n$, $\mu_n = J_{r_n} w_n$ for all $n \geq 0$, $p \in \Omega$ and by Lemma 2.14, we get

$$\begin{aligned}
G(p, Jz_n) &= G(p, JT_{r_n} u_n)\\
&\leq G(p, Ju_n)\\
&= G(p, \beta_n Jw_n + (1-\beta_n)J\mu_n)\\
&= \| p \|^2 - 2\langle p, \beta_n Jw_n + (1-\beta_n)J\mu_n\rangle + \|\beta_n Jw_n\\
&+ (1-\beta_n)J\mu_n \|^2 + 2\sigma f(p)\\
&\leq \| p \|^2 - 2\beta_n\langle p, Jw_n\rangle - 2(1-\beta_n)\langle p, J\mu_n\rangle + \beta_n \| Jw_n \|^2\\
&+ (1-\beta_n) \| J\mu_n \|^2 + 2\sigma f(p)\\
&= \beta_n G(p, Jw_n) + (1-\beta_n)G(p, J\mu_n)\\
&= \beta_n G(p, Jw_n) + (1-\beta_n)G(p, JJ_{r_n} w_n)\\
&= \beta_n G(p, Jw_n) + (1-\beta_n)G(p, Jw_n)\\
&= G(p, Jw_n),
\end{aligned} \tag{11}$$

which implies that

$$G(p, Jz_n) \leq G(p, Jw_n).$$

So, $p \in C_{n+1}$. Therefore by induction $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is well defined. Since $f :\longrightarrow \mathbb{R}$ is convex and lower semi continuous mapping, then it follows from Lemma 2.9 that there exists $q^* \in E^*$ and $\gamma \in \mathbb{R}$ such that

$$f(x) \geq \langle x, q^*\rangle + \gamma, \forall x \in E$$

Now, for $x_n \in E$, we have

$$\begin{aligned}
G(x_n, Jx_0) &= \| x_n \|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\sigma f(x_n)\\
&\geq \| x_n \|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\| + 2\sigma\langle x_n, q^*\rangle + 2\sigma\gamma\\
&= \| x_n \|^2 - 2\langle x_n, Jx_0 - \sigma q^*\rangle + \|x_0\|^2 + 2\sigma\gamma\\
&\geq \| x_n \|^2 - 2 \| x_n \|\| Jx_0 - \sigma q^* \| + \| x_0 \|^2 + 2\sigma\gamma\\
&= (\| x_n \| - \| Jx_0 - \sigma q^* \|)^2 + \| x_0 \|^2 - \| Jx_0 - \sigma q^* \|^2 + 2\sigma\gamma
\end{aligned} \tag{12}$$

then, for each $p \in \Omega \subset C_n$ and $x_n = \Pi^f_{C_n} x_0$, it follows from definition of C_n and (9) that

$$\begin{aligned}
G(p, Jx_0) &\geq G(x_n, Jx_0)\\
&\geq (\| x_n \| - \| Jx_0 - \sigma q^* \|)^2 + \| x_0 \|^2 - \| Jx_0 - \sigma q^* \|^2 + 2\sigma\gamma
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded and so are $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{G(x_n, Jx_0)\}$.
From $x_{n+1} = \Pi^f_{C+1} x_0 \in C_{n+1} \subset C_n$, $x_n = \Pi^f_{C_n} x_0$, and by Lemma 2.11, we have

$$\begin{aligned}
0 &\leq (\| x_{n+1} \| - \| x_n \|)^2\\
&\leq \phi(x_{n+1}, x_n)\\
&\leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0)
\end{aligned} \tag{13}$$

Hence, $\{G(x_n, Jx_0)\}$ is non decreasing. This implies that $\lim\limits_{n\to\infty} G(x_n, Jx_0)$ exists. Now, for any $m > n,\ x_n = \Pi^f_{C_n} x_0,\ \ x_m = \Pi^f_{C_m} x_0 \in C_m \subset C_n$ and by (13), we obtain

$$\phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0).$$

By letting $m, n \longrightarrow \infty$, we get

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0.$$

It follows from Lemma 2.4 that

$$\lim_{n\to\infty} \| x_m - x_n \| = 0.$$

Thus, $\{x_n\}$ is cauchy. Since C is closed subset of Banach space E and C_n is closed and convex. We assume that there exists a point $\hat{x} \in C$ such that

$$\lim_{n\to\infty} x_n = \hat{x}. \tag{14}$$

Also, since $\lim\limits_{n\to\infty} G(x_n, Jx_0)$ exists then by (13), we conclude that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{15}$$

Now, by Lemma 2.4

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{16}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \| Jx_{n+1} - Jx_n \| = 0. \tag{17}$$

By the definition of w_n from (9), we have

$$\| w_n - x_n \| = \| \alpha_n (x_n - x_{n-1}) \| \leq \| x_n - x_{n-1} \| .$$

This implies that

$$\lim_{n\to\infty} \| w_n - x_n \| = 0. \tag{18}$$

Also by (14) and (18), we have

$$\lim_{n\to\infty} w_n = \hat{x}. \tag{19}$$

Since $\{x_n\}$ is bounded, then by Remark 2.5 and (18), we have

$$\lim_{n\to\infty} \phi(w_n, x_n) = 0. \tag{20}$$

Using (16) and (18), we get that

$$\lim_{n\to\infty} \| x_{n+1} - w_n \| = 0. \tag{21}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we conclude that

$$\lim_{n\to\infty} \| Jx_{n+1} - Jw_n \| = 0. \tag{22}$$

Also by Remark 2.5 and (21), we get

$$\lim_{n\to\infty}\phi(x_{n+1}, w_n) = 0. \tag{23}$$

Now, from the definition of C_{n+1} in (9) and $x_{n+1} = \Pi^f_{C_{n+1}} x_0$, we have

$$G(x_{n+1}, Jz_n) \le G(x_{n+1}, Jw_n).$$

This is equivalent to

$$\begin{aligned} \| x_{n+1} \|^2 &- 2\langle x_{n+1}, Jz_n\rangle + \|z_n\|^2 + 2\sigma f(x_{n+1}) \\ &\le \| x_{n+1} \|^2 - 2\langle x_{n+1}, Jw_n\rangle + \|w_n\|^2 + 2\sigma f(x_{n+1}). \end{aligned}$$

Implies that

$$\begin{aligned} \| x_{n+1} \|^2 &- 2\langle x_{n+1}, Jz_n\rangle + \|z_n\|^2 \\ &\le \| x_{n+1} \|^2 - 2\langle x_{n+1}, Jw_n\rangle + \|w_n\|^2, \end{aligned}$$

this gives

$$\phi(x_{n+1}, z_n) \le \phi(x_{n+1}, w_n).$$

Therefore by using (23), we obtain

$$\lim_{n\to\infty}\phi(x_{n+1}, z_n) = 0.$$

It follows from Lemma 2.4 that

$$\lim_{n\to\infty} \| x_{n+1} - z_n \| = 0. \tag{24}$$

By J is uniformly norm-to-norm continuous on bounded subsets of E, we get that

$$\lim_{n\to\infty} \| Jx_{n+1} - Jz_n \| = 0. \tag{25}$$

Taking into account that

$$\| x_n - z_n \| \le \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| . \tag{26}$$

Putting (16) and (24) in (26), we obtain

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \tag{27}$$

Since $x_n \longrightarrow \hat{x}$ (as $n \longrightarrow \infty$) and by (27), we conclude that

$$\lim_{n\to\infty} z_n = \hat{x}. \tag{28}$$

Using (18) and (27), we obtain

$$\lim_{n\to\infty} \| w_n - z_n \| = 0. \tag{29}$$

Since J is uniformly continuous on bounded subset of E, we have

$$\lim_{n\to\infty} \| Jw_n - Jz_n \| = 0. \tag{30}$$

Also, from the definition of C_{n+1}, we have

$$G(x_{n+1}, Ju_n) \leq G(x_{n+1}, Jw_n).$$

Which is equivalent to

$$\begin{aligned} \| x_{n+1} \|^2 &- 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 + 2\sigma f(x_{n+1}) \\ &\leq \| x_{n+1} \|^2 - 2\langle x_{n+1}, Jw_n\rangle + \|w_n\|^2 + 2\sigma f(x_{n+1}). \end{aligned}$$

This implies that

$$\begin{aligned} \| x_{n+1} \|^2 &- 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &\leq \| x_{n+1} \|^2 - 2\langle x_{n+1}, Jw_n\rangle + \|w_n\|^2, \end{aligned}$$

we get

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, w_n).$$

Using (23), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

By Lemma 2.4 we conclude that

$$\lim_{n\to\infty} \| x_{n+1} - u_n \| = 0. \tag{31}$$

By J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \| Jx_{n+1} - Ju_n \| = 0. \tag{32}$$

By triangular inequality, we have

$$\| x_n - u_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - u_n \| . \tag{33}$$

Also, putting (16) and (31) in (33), we conclude that

$$\lim_{n\to\infty} \| x_n - u_n \| = 0. \tag{34}$$

Since $x_n \longrightarrow \hat{x}$ (as $n \longrightarrow \infty$), it follows from (34) that

$$\lim_{n\to\infty} u_n = \hat{x}. \tag{35}$$

Since J is uniformly norm-to- norm continuous on bounded sets, it also follows from (34) that

$$\lim_{n\to\infty} \| Jx_n - Ju_n \| = 0. \tag{36}$$

Using (25) and (32), we obtain

$$\lim_{n\to\infty} \| Jz_n - Ju_n \| = 0. \tag{37}$$

From (11), we have

$$\begin{aligned} G(p, Jz_n) &= G(p, JT_{r_n} u_n) \\ &\leq G(p, Ju_n) \\ &\leq \beta_n G(p, Jw_n) + (1 - \beta_n) G(p, J\mu_n), \end{aligned}$$

this implies that

$$G(p, J\mu_n) \geq \frac{1}{1-\beta_n}(G(p, Jz_n) - \beta_n G(p, Jw_n))$$

Furthermore, by Lemma 2.14, we notice that

$$\begin{aligned}
\phi(\mu_n, w_n) &= \phi(J_{r_n} w_n, w_n) \\
&\leq G(p, Jw_n) - G(p, JJ_{r_n} w_n) \\
&= G(p, Jw_n) - G(p, J\mu_n) \\
&\leq G(p, Jw_n) - \frac{1}{1-\beta_n}(G(p, Jz_n) - \beta_n G(p, Jw_n)) \\
&= \frac{1}{1-\beta_n}(G(p, Jw_n) - G(p, Jz_n)) \\
&= \frac{1}{1-\beta_n}(\| w_n \|^2 - \| z_n \|^2 - 2\langle p, Jw_n - Jz_n \rangle) \\
&\leq \frac{1}{1-\beta_n}(\| w_n \|^2 - \| z_n \|^2 + 2|\langle p, Jw_n - Jz_n \rangle|) \\
&\leq \frac{1}{1-\beta_n}((\| w_n - z_n \|)(\| w_n + z_n \|) + 2 \| p \| \| Jw_n - Jz_n \|)
\end{aligned} \tag{38}$$

Since $\liminf\limits_{n\to\infty}(1-\beta_n) > 0$, using (29) and (30), we conclude that

$$\lim_{n\to\infty} \phi(\mu_n, w_n) = 0. \tag{39}$$

Now, by Lemma 2.4, we have

$$\lim_{n\to\infty} \| w_n - \mu_n \| = 0. \tag{40}$$

Since J is uniformly norm-to- norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \| Jw_n - J\mu_n \| = 0. \tag{41}$$

Also, using (19) and (40) we obtain $\mu_n \longrightarrow \hat{x}$ (as $n \to \infty$).

Now, from $r_n \geq a$, $\mu_n = J_{r_n} w_n$ and by (41), we have

$$\lim_{n\to\infty} \frac{1}{r_n} \| Jw_n - J\mu_n \| = 0. \tag{42}$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \| S_{r_n} w_n \| &= \lim_{n\to\infty} \frac{1}{r_n} \| Jw_n - JJ_{r_n} w_n \| \\
&= \lim_{n\to\infty} \frac{1}{r_n} \| Jw_n - J\mu_n \| \\
&= 0.
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup \hat{x}$ as $m \longrightarrow \infty$. Furthermore, it follows from (18), (27), (34), and (40) that $w_{n_m} \rightharpoonup \hat{x}$, $z_{n_m} \rightharpoonup \hat{x}$, $u_{n_m} \rightharpoonup \hat{x}$, and $\mu_{n_m} \rightharpoonup \hat{x}$ (as $m \longrightarrow \infty$) respectively. By the fact that S is monotone and $(\varpi, \varpi^*) \in S$, it follows from Lemma 2.13 that

$$\langle \varpi - \mu_{n_m}, \varpi^* - S_{r_{n_m}} w_{n_m} \rangle \geq 0, \quad \forall n \geq 0.$$

Taking the limit as $m \longrightarrow \infty$, we obtain $\langle \varpi - \hat{x}, \varpi^* \rangle \geq 0$. By the maximality of S, we have

$$\hat{x} \in S^{-1}0.$$

Next, we show that $\hat{x} \in GMEP(\Psi, \Phi, \varphi)$. Since $z_n = T_{r_n} u_n$, It follows from (37) and $r \geq a$ that

$$\lim_{n\to\infty} \| \frac{Jz_n - Ju_n}{r_n} \| = 0 \tag{43}$$

From $z_n = T_{r_n} u_n$, we get

$$\Gamma(z_n, y) + \frac{1}{r_n}\langle y - z_n, Jz_n - Ju_n \rangle \geq 0, \forall y \in C.$$

Replacing n by n_m, it follows from (L_2) that

$$\frac{1}{r_n}\langle y - z_{n_m}, Jz_{n_m} - Ju_{n_m} \rangle \geq -\Gamma(z_{n_m}, y) \geq \Gamma(y, z_{n_m}), \forall y \in C. \tag{44}$$

Letting $m \longrightarrow \infty$ in (44) and by (L_4), we get

$$\Gamma(y, \hat{x}) \leq 0, \forall y \in C.$$

For λ with $0 < \lambda \leq 1$, and $y \in C$, assume that $y_\lambda = \lambda y + (1 - \lambda)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we get that $y_\lambda \in C$ and $\Gamma(y_\lambda, \hat{x}) \leq 0, \forall y \in C$.
By (L_1) and (L_3), we have

$$\begin{aligned} 0 &= \Gamma(y_\lambda, y_\lambda) \\ &\leq \lambda\Gamma(y_\lambda, y) + (1 - \lambda)\Gamma(y_\lambda, \hat{x}) \\ &\leq \lambda\Gamma(y_\lambda, y). \end{aligned}$$

Dividing by λ, we get

$$\Gamma(y_\lambda, y) \geq 0, \forall y \in C.$$

Letting $\lambda \longrightarrow \infty$ and by (L_3), we conclude that

$$\Gamma(\hat{x}, y) \geq 0, \forall y \in C.$$

This implies that $\hat{x} \in GMEP(\Psi, \Phi, \varphi)$. Hence $\hat{x} \in \Omega$.
Next, we show that $\hat{x} = \Pi^f_C x_0$. Setting $t^* = \Pi^f_C x_0$, and by Lemma 2.10, it follows that $\Pi^f_C x_0$ is single valued. From the fact that $x_n = \Pi^f_C x_0$ and $\Omega \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(t^*, Jx_0)$$

Also, from the definition of G and f, we have that for each x, $G(y, Jx)$ is convex and lower semi continuous with respect to y. Now from the fact that norm is weakly lower semi continuous, we get

$$\begin{aligned} G(\hat{x}, Jx_n) &= \| \hat{x} \|^2 - 2\langle \hat{x}, Jx_0 \rangle + \|x_0\|^2 + 2\sigma f(\hat{x}) \\ &\leq \liminf_{m\to\infty}(\| x_{n_m} \|^2 - 2\langle x_{n_m}, Jx_0 \rangle + \|x_0\|^2 + 2\sigma f(x_{n_m})) \\ &\leq \liminf_{m\to\infty} G(x_{n_m}, Jx_0) \\ &\leq G(t^*, Jx_0) \end{aligned} \tag{45}$$

But

$$G(t^*, Jx_0) \leq G(z, Jx_0), \forall z \in \Omega \tag{46}$$

Therefore, $G(\hat{x}, Jx_0) = G(t^*, Jx_0)$. Now, from the uniqueness of $\Pi_C^f x_0$, we have $\hat{x} = t^*$. Finally, we show that $x_{n_m} \longrightarrow \hat{x}$ (as $n \to \infty$). It follows from (45) and (46) that

$$\lim_{n\to\infty} G(x_{n_m}, Jx_0) = \lim_{n\to\infty} G(\hat{x}, Jx_0)$$

Thus, $\| x_{n_m} \| \longrightarrow \| \hat{x} \|$, as $m \longrightarrow \infty$. Since $x_{n_m} \rightharpoonup \hat{x}$ (as $m \longrightarrow \infty$), it follows from Lemma 2.12 that $x_{n_m} \longrightarrow \hat{x}$ (as $m \longrightarrow \infty$). Hence $x_n \longrightarrow \Pi_\Omega^f x_0$. This completes the proof.

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex real Banach space and C be a nonempty closed and convex subset of E. Let $\Psi : C \times C \longrightarrow \mathbb{R}$ be a bi function which satisfies conditions $(L_1) - (L_4)$, $\Phi : C \longrightarrow E^*$ be continuous and monotone, and $\varphi : C \longrightarrow \mathbb{R}$ be a lower semi-continuous and convex function. Let $f : E \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset int(D(f))$, where $D(f)$ is the domain of f. Let $S \subset E \times E^*$ be a maximal monotone operator satisfying $D(S) \subset C$ and $J_{r_n} = (J + r_n S)^{-1} J$, for all $r_n > 0$. Assume that $\Omega := GMEP(\Psi, \Phi, \varphi) \cap S^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0 = E; \\ u_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JJ_{r_n} x_n); \\ z_n \in C \text{ such that } \Psi(z_n, y) + \langle \Phi z_n, y - z_n \rangle + \varphi(y) - \varphi(z_n) \\ +\dfrac{1}{r_n}\langle y - z_n, Jz_n - Ju_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : G(z, Jz_n) \leq G(z, Jx_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \ \forall n \in \mathbb{R} \cup \{0\} \end{cases}$$

where β_n is a sequence in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\liminf_{n\to\infty}(1 - \beta_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$. The sequence $\{x_n\}$ converges strongly to $\Pi_\Omega^f x_0$, where Π_Ω^f is the generalized f- projection of E onto Ω.

Corollary 3.3. *Let E be a uniformly smooth and uniformly convex real Banach space and C be a nonempty closed and convex subset of E. Let $f : E \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset int(D(f))$, where $D(f)$ is the domain of f. Let $S \subset E \times E^*$ be a maximal monotone operator satisfying $D(S) \subset C$ and $J_{r_n} = (J + r_n S)^{-1} J$, for all $r_n > 0$. Assume that $S^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0 = E; \\ w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ u_n = J^{-1}(\beta_n Jw_n + (1-\beta_n) JJ_{r_n} w_n); \\ C_{n+1} = \{z \in C_n : G(z, Jz_n) \leq G(z, Jw_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \ \forall n \in \mathbb{R} \cup \{0\} \end{cases}$$

where $\alpha_n \subset (0, 1)$, β_n is a sequence in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\liminf_{n\to\infty}(1 - \beta_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$. The sequence $\{x_n\}$ converges strongly to $\Pi_{S^{-1}0}^f x_0$, where Π_Ω^f is the generalized f- projection of E onto Ω.

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex real Banach space and C be a nonempty closed and convex subset of E. Let $S \subset E \times E^*$ be a maximal monotone operator satisfying $D(S) \subset C$ and $J_{r_n} = (J + r_n S)^{-1} J$, for all $r_n > 0$. Assume that $S^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0 = E; \\ w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ u_n = J^{-1}(\beta_n Jw_n + (1-\beta_n) JJ_{r_n} w_n); \\ C_{n+1} = \{z \in C_n : \phi(z, z_n) \leq \phi(z, w_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \in \mathbb{R} \cup \{0\} \end{cases}$$

where $\alpha_n \subset (0, 1)$, β_n is a sequence in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Assume that $\liminf_{n\to\infty}(1 - \beta_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$. The sequence $\{x_n\}$ converges strongly to $\Pi_{S^{-1}0} x_0$, where Π_Ω is the generalized projection of E onto Ω.

4. Application

In this section, we present some applications of theorem 3.1 as follows:

4.1. *Maximal monotone operator and system of equilibrium problems.*

By setting $\Phi \equiv 0, \varphi \equiv 0$ in theorem 3.1, the sequence defined in theorem 3.1 converges strongly to $\Pi^f_\Omega x_0$, where $\Omega := EP(\Psi) \cap S^{-1}0$ and $EP(\Psi)$ is the set of solution of the equilibrium problem for Ψ.

4.2. *Maximal monotone operator and system of convex optimization problems.*

By setting $\Psi \equiv 0, \Phi \equiv 0$ in theorem 3.1, the sequence defined in theorem 3.1 converges strongly to $\Pi^f_\Omega x_0$, where $\Omega := CMP(\varphi) \cap S^{-1}0$ and $CMP(\varphi)$ is the set of solution of the convex optimization problem for φ.

4.3. *Maximal monotone operator and system of variational inequalities problems.*

By setting $\Psi \equiv 0, \varphi \equiv 0$ in theorem 3.1, the sequence defined in theorem 3.1 converges strongly to $\Pi^f_\Omega x_0$, where $\Omega := VIP(C, \Phi) \cap S^{-1}0$ and $VIP(C, \Phi)$ is the set of solution of variational inequality problem for Φ over C.

5. Conclusion

Theorem 9 improves the result of Siwaporn Saewan [23]. Since our result involved maximal monotone operator and generalized mixed equilibrium problems as against only maximal monotone operator. Also, our iterative scheme incoorperates inertial term that speed the convergence rate of iterative sequence. Furthermore, this work improves the work of Chidume *et al.* [5] by incoorperating the system of generalized mixed equilibrium problems in the iterative scheme and also extends the work from generalized projection to generalized f- projection. Because of the slight modification of the iterative scheme by incoorperating the inertial term, our result in this paper extends the work of Siwaporn and kumam [20] from the system of generalized Kly Fan inequality to the system generalized mixed equilibrium problems. We apply this result to the system of equilibrium problems, convex optimization problems and variational inequality problems in Banach spaces. Finally, our theorem improves and extends the main results of Siwaporn Saewan [23], Chidume *et al.* [5], and Siwaporn and kumam [20] and many results in the literature.

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