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Convergence and Stability of the Ishikawa Iterative Process for a class of φ-quasinonexpansive Mappings

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Abstract

The paper analyzes the convergence of Ishikawa iteration to the fixed point of a class of φ-quasinonexpansive mappings in uniformly convex Banach spaces, as well as the stability of the Ishikawa iteration used in approximating the fixed point. The work not only confirmed Ishikawa iteration's convergence and stability to the fixed point of φ-quasinonexpansive mappings, but it also pointed the way for future research in the estimate of fixed points of φ-quasinonexpansive mappings.

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1. Introduction and preliminaries

If $Tx = x$, then X is a nonempty set with a self-map $T : X \longrightarrow X$, and a point $x \in X$ is said to be a fixed point of T. The set of positive real numbers is designated by $\mathbb{R}_+$ throughout this work, while the set $\{x \in X : Tx = x\}$ of fixed points of a mapping T is indicated by F_T. Let E be a Banach space that is uniformly convex and D be a closed convex subset of E. Let $T : D \longrightarrow D$ be a nonlinear mapping; a mapping T is said to be contractive if there is a constant $k \in [0, 1)$ such that $\|Ta - Tb\| \leq k\|a - b\| \forall a, b \in D$. T is said to be nonexpansive if $\exists$ a constant $k = 1$ such that $\|Ta - Tb\| \leq \|a - b\|$ for all $a, b \in D$. In 1970, Dotson [1] considered a certain class of mapping known as quasi-nonexpansive mappings which was also studied by Zamfirescu [2], Ciric [3], Berinde [4], Olatinwo [5] and many other authors. A mapping T is called quasi-nonexpansive if at least there is a fixed point p such that $\|Ta - p\| \leq \|a - p\| \ \forall \ a \in D$ and $p \in F_T$.

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Example 1.1. Let $V = \mathbb{R}$ and let T be defined as follows:

$$T(0) = 0,$$

$$Ta = \frac{a}{2}\sin\left(\frac{1}{a}\right), \quad for\ a \neq 0$$

The only fixed point of T is 0, since if $a \neq 0$ and $Ta = a$, then

$$a = \frac{a}{2}\sin\left(\frac{1}{a}\right), \quad or\ 2 = \sin\left(\frac{1}{a}\right)$$

which is not possible. Then, suppose T is quasi-nonexpansive since $a \in D$, $p = 0$ we have

$$\|Ta - p\| = \|Ta - 0\| = \|a\|\left|\sin\frac{1}{a}\right| = \|a\| = \|a - p\|.$$

However, T is not a nonexpansive mapping. This can be seen choosing $a = \frac{2}{\pi}$ and $b = \frac{2}{3\pi}$, then

$$\|Ta - Tb\| = \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{3\pi}\sin\frac{3\pi}{2} = \frac{2}{\pi} + \frac{2}{3\pi} = \frac{8}{3\pi},$$

whereas,

$$\|a - b\| = \frac{8}{3\pi}.$$

This demonstrates that nonexpansive mappings are properly included in the class of quasi-nonexpansive mappings. If nonexpansive mappings with the set of fixed points $F_T = \emptyset$ are quasi-nonexpansive and linear quasi-nonxpansive mappings are nonexpansive, we can conclude from the example above that nonlinear continuous quasi-nonexpansive mappings that are not nonexpansive exist.

2. Iterative processes

Let E be a uniformly convex Banach space and D be a closed convex subset of E and $T : D \to D$ a self-mapping. We can examine the sequence $\{a_n\}_{n=0}^{\infty}$ generated by an iteration procedure. Let $a_0 \in D$,

$$a_{n+1} = Ta_n, \quad n = 0, 1, \cdots \tag{1}$$

The sequence generated by (1) is reffers to successive approximation also known as Picard iteration. In the scenario where T is nonexpansive, and the contractive requirements are slightly weak, the sequential approximation does not have to converge to a fixed point of T.
The Mann iterative process is define as

$$a_{n+1} = (1 - \alpha_n)a_n + \alpha_n Ta_n, \quad n = 0, 1, 2, 3, \cdots \tag{2}$$

If T is constant and the Mann iterative process converges, it does so to a fixed point of T, but if T is not continuous, there is no guarantee that the Mann iterative process will converge to a fixed point of T.
Recently, R. Jahed et al. [16] investigate the iterations of ϕ-quasinonexpansive mappings and their applications in Banach spaces in their paper. First, they show that the sequence generated by the hybrid proximal point method has strong convergence to a common fixed point of a family of ϕ-quasinonexpansive mappings as well as applying their main results to equilibrium problems. Atailia et al. [17] lately combined Hardy and Rogers [18] type nonexpansive mappings and Suzuki generalized nonexpansive mapping to introduce a new class of mappings known as generalized contractions of Suzuki type mappings. They were able to obtain some fixed point results for their new class of nonexpansive type mappings. R. Pant et al. [19] consider the Halpern iteration in 2021 for finding a common fixed point of a nonexpansive type semigroup and a countable family of mappings satisfying condition (E). As a result, the findings in [16, 17] have been expanded, generalized, and complemented.

Ariza-Ruize [6] showed that the mappings of a convex metric space are in class of φ-Quasinonexpansive mappings for Mann iterative process. The algorithm presented by Ishikawa (1974) is as follows:

$$\begin{cases} b_n = (1-\beta_n)a_n + \beta_n T a_n \ , \\ a_{n+1} = (1-\alpha_n)a_n + \alpha_n T b_n \ . \end{cases} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] satisfying the successive parameters:
1. $0 \leq \alpha_n \leq \beta_n \leq 1$;
2. $\lim_{n\to\infty} \beta_n = 0$;
3. $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$
which converges to a fixed point of a Lipschitz Pseudocontrative self-mapping T on compact, convex subset of Hilbert space C. We consequently, regard Ishikawa iteration method as a two step Mann iteration with two different parameter sequences.
The iterative (3) above has been studied widely for both metric and Banach spaces for instance, Osiliki [12], Olatinwo [14], Chidume [15] among others.

Definition 2.1. *(Convex Metric Space) A convex metric space $(E, d, \oplus)$ is a metric space (E, d) together with a convexity mapping $\oplus : E \times E \times [0, 1] \longrightarrow E$ satisfying*

$$d(r, (1-\lambda)p \oplus \lambda q) \leq (1-\lambda)d(r, p) + \lambda d(r, q) \tag{4}$$

for all $p, q, r \in E, \lambda \in [0, 1]$.

Takahashi [7] in 1970 introduced a type of metric spaces called convex metric spaces, this class of metric spaces has brought a lot improvement and generality of fixed point results on Banach spaces. For example, Zhao-hong Sun [8], Ciric [9] for non self mapping.

Theorem 2.1. *(For more information, see [6]). Consider the convex subset C of a convex metric space $(E, d, \oplus)$. Assume $T : C \longrightarrow C$ is a φ-quasinonexpansive mapping with $\varphi \in \Phi$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a real sequence in the range $[0; 1]$ that converges to a positive real value. The sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by the Mann iteration process then converges to the unique fixed point of T for any $a_0 \in E$.*

The purpose of this study is to generalize Theorem 1.1 by utilizing a better Ishikawa iteration method to show both the fixed point and stability results in uniformly convex Banach spaces with the condition $\varphi(t) \leq t \ \forall \ t \geq 0$ uniformly convex Banach spaces.

Definition 2.2. If there is a $\delta(\epsilon) > 0$ for any, $0 < \epsilon \leq 2$, then the Banach space E is said to be uniformly convex when

$$||\mathrm{u}|| = ||\mathrm{v}|| = 1, \ ||\mathrm{u} - \mathrm{v}|| \geq \epsilon, \text{it implies that } \left\| \frac{\mathrm{u}+\mathrm{v}}{2} \right\| \geq 1 - \delta(\epsilon). \tag{5}$$

Let E be a Banach space with a component of itself such as $E \geq 2$. The function $\delta_E(\epsilon) : [0, 2) \longrightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - ||\frac{1}{2}(a = b)|| : ||a|| = 1, ||b|| = 1, \epsilon = ||a - b||\}$$

is called modulus of E. If $\delta_E(\epsilon) > 0 \ \forall \ \epsilon \in [0, 2)$ then, Banach space E is uniformly convex. We define Opial condition as follows for any sequence $\{b_n\} \in V$ of a normed space $(V, ||.||)$ thus $b_n \rightharpoonup b_0$ it follows that $\forall \ c \in V, \ c \neq b_0$,

$$\limsup_{n\longrightarrow\infty} ||b_n - b_0|| < \limsup_{n\longrightarrow\infty} ||b_n - c||,$$

which can be shown from the extraction of those connected subsequences if the lower limits and upper limits are interchange in the definition above.

For the sake of this paper we want to generalized quasi-nonexpansive mappings by introducing $\varphi \in [0,1]$ and $\lambda \in [0, \frac{1}{2}]$ of reals to the quasi-nonexpnsive mapping $||Ta - p|| \leq ||a - p||$ which gives a class of mappings below

$$||\mathrm{Ta} - \mathrm{Tp}|| \leq (1-\varphi)(||\mathrm{a} - \mathrm{p}||) + \lambda(||\mathrm{a} - \mathrm{Tp}|| + ||\mathrm{p} - \mathrm{Ta}||) \tag{6}$$

for all subsets $a \in D$ of uniformly convex Banach space E and $p \in F_T$ on D, which happens to be the focus of our research. The mapping T that satisfies (6) is known as generalized quasinonexpansive mapping.

Example 2.1. Let D be the real line mapping $T : D \longrightarrow D$ defined by

$$Ta = \begin{cases} \left(\frac{a}{4}\cos\left(\frac{1}{a}\right)\right) & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Suppose $a = 2$, $p \in F_T$ then $||Ta - Tp|| = ||\left(\frac{2}{4}\cos\left(\frac{1}{2}\right) - p\right)||$
$$\leq (1-\varphi)(||2-0||) + \lambda(||2-0|| + ||0 - (\tfrac{2}{4}\cos\left(\tfrac{1}{2}\right))||)$$

with $\varphi \in [0,1]$ and $\lambda \in [0, \frac{1}{2}]$ we have
$$\leq \left(\tfrac{1}{2}\right)(||2-0|| + ||0 - (\tfrac{2}{4}\cos\left(\tfrac{1}{2}\right))||)$$
$$\leq 1 + \tfrac{2}{4}\cos\left(\tfrac{1}{2}\right).$$
Hence, T is a generalized quasinonexpansive mapping. Clearly, the class includes qusinonexpansive mappings if $\varphi = \lambda = 0$.
Similarly, the mapping satisfies condition C i.e,
$\frac{1}{2}||Ta - a|| \leq ||a - p|| \Longrightarrow ||Ta - Tp|| \leq ||a - p||$.

The following lemmas and definition will be a useful tools for this research work.

Lemma 2.1. Let D be a nonempty subset of a uniformly Banach space E. A mapping $T : D \longrightarrow D$ satisfies the generalized quasinonexpansive if $p \in F_T$ on D and for all $a \in D$,
$||Tp - Ta|| \leq ||p - a||$.
Proof
$$||Ta - Tp|| \leq (1-\varphi)(||a-p||) + \lambda(||a - Tp|| + ||p - Ta||)$$
$$= (1-\varphi)(||a-p||) + \lambda(||a-p|| + ||p - Ta||)$$
$$||Ta - Tp|| \leq \frac{1-\varphi+\lambda}{1-\lambda}||a-p|| \leq ||a-p||.$$

Lemma 2.2. [10] Let $a, b \in X$. If $||a|| \leq 1$, $||b|| \leq 1$ *and* $||a - b|| \geq \epsilon > 0$, then

$$||\lambda a + (1-\lambda)b|| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon) \quad for \quad 0 \leq \lambda < 1.$$

Lemma 2.3. [11] Let $\alpha_n \geq 0, \sigma_n \geq 0$ be such that

$$\alpha_{n+1} \leq (1+\alpha_n)a_n + \sigma_n. \tag{7}$$

If, (a) $\sum_{n=1}^{\infty} \alpha_n < \infty$; (b) $\sum_{n=1}^{\infty} \sigma_n < \infty$; and (c) $\lim_{n\longrightarrow\infty} \inf a_n = 0$, then $\lim_{n\longrightarrow\infty} a_n = 0$.

Definition 2.3. Harder and Hicks [13]. Let $T : X \longrightarrow X$ be a map and (X, d) be a complete metric space. Assume that the set of fixed points of T is $F_T = \{p \in X \mid Tp = p\}$. Let $\{b_n\}_{n=0}^{\infty} \subset X$ denote the sequence generated by an iterative operation involving S, defined by

$$b_{n+1} = f(T, b_n), \quad b_0 \in X, \ n = 0, 1, 2, ..., .$$

The first approximation is $b_0 \in X$, and function f is some function. Assume that the sequence $\{b_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{c_n\}_{n=0}^{\infty} \subset X$ and set $\varepsilon_n = d(c_{n+1}, f(T, c_n))$, $n = 0, 1, 2, ...$, and the iterative technique above is T- stable or stable with regard to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} c_n = p$.

We now proceed by showing the major sequel of this paper.

3. Main Results

Theorem 3.1. Let E be a non-empty closed convex subset of E, and D be a uniformly convex Banach space. Assume that the mapping $T : D \longrightarrow D$ is a generalized quasinonexpansive mapping. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be sequences in interval $[0, 1]$ that converge to some positive real values. Then the sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} a_1 \in D, \\ a_{n+1} = (1-\alpha_n)a_n + \alpha_n T b_n, \\ b_n = (1-\beta_n)a_n + \beta_n T a_n, \ n \in \mathbb{N}. \end{cases} \tag{8}$$

converges to the unique fixed point of T.

Proof
Let $p \in F_T$ denote the fixed point of T, employing (8) and (6),

$$\begin{aligned} \|b_n - p\| &= \|(1-\beta_n)a_n + \beta_n T a_n - p\| \\ &= \|(1-\beta_n)(a_n - p) + \beta_n(T a_n - p)\| \\ &\le (1-\beta_n)\|a_n - p\| + \beta_n \|T a_n - p\| \\ &\le (1-\beta_n)\|a_n - p\| + \beta_n[(1-\varphi)(\|a - p\|) + \lambda(\|a - T p\| + \|p - T a\|)] \end{aligned}$$

introducing Lemma (1.1) then, we have

$$\|b_n - p\| \le (1-\beta_n)\|a_n - p\| + \beta_n\|a_n - p\|$$

$$\|b_n - p\| \le \|a_n - p\| \tag{9}$$

But

$$\begin{aligned} \|a_{n+1} - p\| &= \|(1-\alpha_n)a_n + \alpha_n T b_n - p\| \\ &= \|(1-\alpha_n)(a_n - p) + \alpha_n(T b_n - p)\| \\ &\le (1-\alpha_n)\|a_n - p\| + \alpha_n\|T b_n - p\| \end{aligned} \tag{10}$$

Since T is a generalized quasinonexpansive
then,

$$\begin{aligned} \|T b_n - p\| &\le (1-\varphi)(\|b_n - p\|) + \lambda(\|b_n - T p\| + \|p - T b_n\|) \\ &= (1-\varphi)(\|b_n - p\|) + \lambda(\|b_n - p\| + \|p - T b_n\|) \\ &\le \left(\frac{1-\varphi+\lambda}{1-\lambda}\right)\|b_n - p\| \le \|b_n - p\| \end{aligned}$$

$\|T b_n - p\| \le \|b_n - p\|$
substituting equation (9) gives

$$\|T b_n - p\| \le \|a_n - p\| \tag{11}$$

substituting equation (11) into (10) we have

$$\|a_{n+1} - p\| \le (1-\alpha_n)\|a_n - p\| + \alpha_n\|a_n - p\|$$

$$\|a_{n+1} - p\| \le \|a_n - p\| \tag{12}$$

This affirms that the sequence $\{\|a_n - p\|\}$ is a monotone, non-increasing sequences that converges to a real number say L. If $L > 0$,
then $\|a_n - T b_n\| \le \|a_n - p\| + \|p - T b_n\| \le \|a_n - p\| + \|a_n - p\| \le 2\|a_n - p\|$

Therefore, $\{||a_n - Tb_n||\}$ is bounded. As a result, alongside Bolzano Wieltrass theorem, there exists a convergent subsequence $\{||a_{ni} - Tb_{ni}||\}$ of $\{||a_n - Tb_n||\}$ thus $\lim_{n\to\infty} ||a_{n_i} - Tb_{n_i}|| = \lambda$ especially for $\lambda \in (0, \infty)$.

Set $a = \dfrac{a_n - p}{||a_n - p||}$ and $b = \dfrac{Tb_n - p}{||a_n - p||}$.

Then we have $\quad a - b = \dfrac{a_n - Tb_n}{||a_n - p||}$.

Thus, $||a|| = 1$, $||b|| \leq 1$ and

$$||a - b|| = \frac{||a_n - Tb_n||}{||a_n - p||} \geq \frac{||a_n - Tb_n||}{||a_1 - p||} = \epsilon$$

By the Lemma (1.2), substituting a and b we have,

$$\left\|\alpha_n \frac{(Tb_n - p)}{||a_n - p||} + (1 - \alpha_n)\frac{(a_n - p)}{||a_n - p||}\right\| \leq 1 - 2\alpha_n(1 - \alpha_n)\delta\left(\frac{||a_n - Tb_n||}{||a_1 - p||}\right),$$

such that $0 < \delta \leq 1$

$$||\alpha_n(a_n - p) + (1 - \alpha_n)(Tb_n - p)|| \leq ||a_n - p||\left[1 - 2\alpha_n(1 - \alpha_n)\delta\left(\frac{||a_n - Tb_n||}{||a_1 - p||}\right)\right]$$

By induction,

$$\begin{aligned}
||a_{n+1} - p|| &\leq 1 - 2\alpha_n(1 - \alpha_n)\delta\left(\frac{||a_n - Tb_n||}{||a_1 - p||}\right).||a_n - p|| \\
&\leq 1 - 2\alpha_n(1 - \alpha_n)\delta\left(\frac{||a_n - Tb_n||}{||a_1 - p||}\right) \\
&.\left(1 - 2\alpha_{n-1}(1 - \alpha_{n-1})\delta\left(\frac{||a_{n-1} - Tb_{n-1}||}{||a_1 - p||}\right).||a_{n-1} - p||\right) \\
&\leq 1 - 2\alpha_n(1 - \alpha_n)\delta\left(\frac{||a_n - Tb_n||}{||a_1 - p||}\right).(1 - 2\alpha_{n-1}(1 - \alpha_{n-1})\delta) \\
&\left(\frac{||a_{n-1} - Tb_{n-1}||}{||a_1 - p||}\right)...\left(1 - 2\alpha_1(1 - \alpha_1)\delta\left(\frac{||a_1 - Tb_1||}{||a_1 - p||}\right).||a_1 - p||\right)
\end{aligned}$$

$$||a_{n+1} - p|| \leq M\prod_{i=1}^{n}\left[1 - 2\alpha_i(1 - \alpha_i)\delta\left(\frac{||a_i - Tb_i||}{||a_1 - p||}\right)\right] \tag{13}$$

where $M = ||a_1 - p||$

Considering δ as a monotone increasing and

$$1 - 2\alpha_i(1 - \alpha_i)\delta\left(\frac{||a_i - Tb_i||}{M}\right) \in (0, 1)$$

for all i, since $\lim_{i\longrightarrow\infty} ||a_{n_i} - Tb_{n_i}|| = \lambda$, and $M = ||a_1 - p|| \ \forall \ n \geq \mathbb{N}$, it follows from (13)

$$||a_{n_i} - p|| \leq M\prod_{i=1}^{n}\left[1 - 2\alpha_i(1 - \alpha_i)\delta\left(\frac{||a_i - Tb_i||}{||a_1 - p||}\right)\right]\left[1 - 2\alpha(1 - \alpha)\delta\left(\frac{\beta}{M}\right)\right]^{n_i - N}$$

As $i \longrightarrow \infty$, $||a_{n_i} - p|| \longrightarrow 0$ which implies $a_{n_i} \longrightarrow p$.

From (9), $||b_n - p|| \leq ||a_n - p||$, it implies $||b_{n_i} - p|| \longrightarrow 0$,

then $b_{n_i} \longrightarrow p$.

Also, since from (11) we have$||Tb_n - p|| \leq ||a_n - p||$, it follows that $||Tb_{n_i} - p|| \longrightarrow 0$ and therefore $Tb_{n_i} \longrightarrow p$.

Since $\lim_{i\longrightarrow\infty} Tb_{n_i} = p \in E$ and $\lim_{i\longrightarrow\infty} b_{n_i} = p$. Then it follows,

$$\begin{aligned}
||Tp - p|| &\leq ||Tp - Tb_{n_i}|| + ||Tb_{n_i} - p|| \\
&\leq ||p - b_{n_i}|| + ||Tb_{n_i} - p||
\end{aligned}$$

since T is continuous on $\mathbb{R},^{+}$ $as\ i \longrightarrow \infty$ then

$$\|Tp - p\| \leq \|p - p\| + \|p - p\|$$
$$\|Tp - p\| \leq 0$$

which implies $Tp = p$ then, p is a fixed point of T.
Since $\{\|a_n - p\|\}$ is monotone non increasing, therefore it is bounded above. Thus, it converges with the $\lim_{i\longrightarrow\infty} a_{n_i} = p$ and $\lim_{i\longrightarrow\infty} a_n = p$ implies that the sequence $\{a_n\}$ converges to a fixed point of T.

Remark 3.1. Theorem (1.2) is an extension of Theorem (1.1).

The stability of the Ishikawa algorithm using φ-quasinonexpansive mapping in uniformly convex Banach spaces is demonstrated in the following theorem.

Theorem 3.2. Let E be a Banach space that is uniformly convex and D be a non-empty closed convex subset of E. Let $T : D \longrightarrow D$ be a generalized quasinonexpansive mapping with $\varphi \in [0, 1]$ and $\lambda \in [0, \frac{1}{2}]$. Let $a_0 \in E$ and the sequence $\{a_n\}_{n=0}^{\infty}$ be defined by

$$\begin{cases} a_1 \in D, \\ a_{n+1} = (1-\alpha_n)a_n + \alpha_n T b_n, \\ b_n = (1-\alpha_n)a_n + \beta_n T a_n,\ n \in \mathbb{N}. \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \subset [0, 1]$. The Ishikawa iterative process then becomes T- stable.

Proof
Following the definition of Ishikawa algorithm above and the generalized quasinonexpansiveness of T,
let $c_n = (1-\beta_n)b_n + \beta_n T b_n$ where $a_n = \dfrac{b_n - p}{\|b_n - p\|}$ and $b_n = \dfrac{Tc_n - p}{\|b_n - p\|}$

$$\begin{aligned}
\|a_n\| = \left\|\frac{b_n - p}{\|b_n - p\|}\right\| \leq 1\ and\ \|b_n\| = \left\|\left(\frac{Tc_n - p}{\|b_n - p\|}\right)\right\| &= \frac{\|Tc_n - Tp\|}{\|b_n - p\|} \\
&\leq \frac{(1-\varphi)(\|c_n - p\|) + \lambda(\|c_n - Tp\| + \|p - Tc_n\|)}{\|b_n - p\|} \\
&\leq \frac{\|c_n - p\|}{\|b_n - p\|} = \frac{\|(1-\beta_n)b_n + \beta_n T b_n - p\|}{\|b_n - p\|} \\
&= \frac{\|(1-\beta_n)(b_n - p) + \beta_n[Tb_n - p]\|}{\|b_n - p\|} \\
&\leq \frac{(1-\beta_n)\|(b_n - p)\| + \beta_n\|Tb_n - p\|}{\|b_n - p\|} \\
&\leq \frac{(1-\beta_n)\|(b_n - p)\| + \beta_n\|b_n - p\|}{\|b_n - p\|} \\
&= \frac{\|b_n - p\|}{\|b_n - p\|} = 1
\end{aligned}$$

Let $\epsilon_n = \|b_{n+1} - f(T, b_n)\| = \|b_{n+1} - (1-\alpha_n)b_n - \alpha_n T c_n\|, \forall\ n = 0, 1, 2, 3, ...,$ and if $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ then, using triangle inequality we get
$\|b_{n+1} - p\| \leq \|b_{n+1} - (1-\alpha_n)b_n - \alpha_n T c_n\| + \|(1-\alpha_n)b_n - \alpha_n T c_n - p\|$

$$= \epsilon_n + \|(1-\alpha_n)(b_n - p) + \alpha_n[Tc_n - p]\|$$

$$= \epsilon_n + \|(1-\alpha_n)a_n\|b_n - p\| + \alpha_n b_n\|b_n - p\|\|$$

$$\leq \epsilon_n + (\|(1-\alpha_n)a_n + \alpha_n b_n\|)\,\|b_n - p\| \tag{14}$$

introducing lemma (1.2) in (14) we have

$$||b_{n+1} - p|| \leq \epsilon_n + [1 - 2\alpha_n(1 - \alpha_n)\delta(\epsilon)]||b_n - p|| \tag{15}$$

wherever $[1 - 2\alpha_n(1 - \alpha_n)\delta(\epsilon)] \in [0, 1]$
By Lemma again (1.2) then

$$\lim_{n \longrightarrow \infty} ||b_n - p|| = 0 \; i.e, \; \lim_{n \longrightarrow \infty} b_n = p$$

Therefore, in the structure of uniformly convex Banach space, the Ishikawa algorithm is stable for the class of φ-quasinonextensive mapping.

4. Conclusion

In conclusion, for the class of φ-quasinonexpansive mappings in uniformly convex Banach space, this paper offered helpful information on the establishment of convergence to the fixed point and stability of the Ishikawa iterative process.

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