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Approximate solution of time-fractional non-linear parabolic equations arising in Mathematical Physics

K. Issa*, R. A. Bello, M. H. Sulaiman

Department of Mathematics and Statistics, Kwara State University, Malete, Kwara State, P. M. B. 1530, Ilorin, Nigeria

Abstract

In this paper, we studied and analysed a new iterative method for solving time-fractional non-linear equations by obtaining approximate solutions to the Allen-Cahn, Newell-Whitehead, and Fisher equations by putting the parameter $\alpha = 1$ and varying the values of γ, ψ, and τ. These three equations were derived from the general non-linear dynamical wave equations when the constants therein assumed certain specific values. Obviously, from the tabulated results, we observed that the approximate solution for each example compares favourably with the existing results in the literature; therefore, the proposed scheme is effective and accurate in solving Allen-Cahn, Newell-Whitehead, and Fisher equations. All the computational work was done using Mathematica, and all the graphs were plotted using MATLAB.

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1. Introduction

One of the most useful mathematical physics models developed in the recent past is the general nonlinear parabolic equation model, which arises in quatum mechanics, plasma physics, and mathematical biology [1–3]. This general dynamical equation produces three major well-known equations with applications in various biological models, quantum mechanics, and plasma physics. These include the Allen-Cahn (AC) equation, the Newell-Whitehead (NW) equation (also known as the Newell-Whitehead-Segel equation), and the Fishers equation [1].

Many brilliant attempts exist in the literature, all aiming at providing the most acceptable solutions to various aspects of the general parabolic non-linear equations. Prominent among them is the modified variational iteration algorithm II that was developed in Ref. [1], Homotopy analysis, and Homotopy-Pade methods employed in Ref. [4] to solve the Newell-Whitehead equation. Li *et al.* [5] investigated the accuracy of two-term time-fractional PDE models using the meshless method. Babolian and Saedian [6] also proposed an analytic approximation approach to the Fishers equation. Sakar *et al.* [7] applied the homotopy perturbation method (HPM) to solve fractional partial differential equations (PDEs) with proportional delay. Natural transform decomposition

*Corresponding author: Tel.: +234-803-655-4437.
Email address: issa.kazeem@kwasu.edu.ng (K. Issa)

method and iterative Shehu transform method were employed to find the numerical solution to the nonlinear time fractional Klein-Gordon equation in Ref. [8]. Issa *et al.* [9, 10] employed shifted Gegenbuer polynomials to find an approximate solution to fractional diffusion equations via the finite difference method and compact finite difference method. The wavelet method was however proposed in Ref. [11] for the solution of some non-linear parabolic equations, while the Legendre wavelet-based approximation method that improved the method proposed in Ref. [11] was presented in Ref. [12]. The non-linear stability of the implicit-explicit methods for the Allen-Cahn equation was elaborately discussed in Ref. [13].

The present work was motivated by the recent work of Hijaz Ahmad and his collaborators reported in Ref. [1]. In their work, the general dynamical parabolic equation was studied with the results presented for the Allen-Cahn, Newell-Whitehead, and Fisher equations. Here in this paper, we studied the same set of equations using the elegant new iterative method (NIM) implemented in the recent work of Akinyemi and his collaborators in Ref. [14].

Many other authors had earlier proposed some iterative methods for solutions to many other families of problems. These include an iterative method that was reported in Ref. [15] for solutions of nonlinear functional equations; NIM was also applied to partial differential equations in Ref. [16]; and a modified iterative method was adopted for solutions of both linear and non-linear Klein-Gordon (KG) equations in Ref. [17]. Other efforts on NIM are Refs. [2, 3, 18–20].

The choice of NIM implemented in Ref. [14] was informed by the simplicity of its implementation and the superiority of the accuracy of the results obtained through it.

The remaining part of this paper is organised as follows: In Section 2, a statement of the problem and method of solution, a general non-linear parabolic dynamical equation, and a description of the iterative method are presented. In Section 3, the implementation of the method described in Section 2 is presented for specific cases of the general equation. The results of numerical experiments are presented in Section 4, with tables representing both $2-D$ and $3-D$ graphical representations of the results. The paper ends in Section 5 with the discussion of results and conclusion.

2. Statement of the problem and method of solution

In this section, the generalised form of the nonlinear parabolic dynamical equation and the iterative method of solution for the three constituents of the equation are discussed.

2.1. General non-linear parabolic dynamical equation

The general nonlinear dynamical wave equation considered in this paper is of the form:

$$D_t^{\alpha} v = v_{xx} + \gamma v + \psi v^{\tau}, \tag{1}$$

where α, γ, ψ, and τ are real constants. It should be noted that α shall be taken as 1 throughout this work.

The nature of Eq. (1) is determined basically by the values assigned to γ, ψ, and τ. For instance, when $\gamma = 1$, $\psi = -1$, and $\tau = 3$, the resulting equation,

$$D_t v = v_{xx} + v - v^3, \tag{2}$$

is called the Allen-Cahn equation. On the other hand, if $\tau = 2$ and $\psi = -\gamma$, the resulting equation,

$$D_t v = v_{xx} + \gamma v - \gamma v^2, \tag{3}$$

is called the Fishers equation. And, when $\psi = -\psi$ and $\tau = 3$, the resulting equation

$$D_t v = v_{xx} + \gamma v - \psi v^3, \tag{4}$$

is called the Newell-Whitehead equation that appears in the discussion of Rayleigh-Benard convection (see Ref. [1] and the references therein).

2.2. Description of the new iterative method (NIM)

Consider a wave equation of the form:

$$v(x,\ t) = \ell(v(x,t)) + N(v(x,t)) + h(x,t), \tag{5}$$

where ℓ and N are linear and non-linear operators, and $h(x,t)$ and $v(x,t)$ are known and unknown functions, respectively.

Let

$$V_m(x,t) = \sum_{i=0}^{m} v_i(x,t), \tag{6}$$

which converges to a unique solution as reported in Refs. [3, 14, 20]. Decomposition of linear and non-linear operators was reported in Refs. [14, 15, 21] as

$$\begin{aligned} \ell(v(x,t)) &= \ell(v_0(x,t)) + \ell(v_1(x,t)) + \cdots + \ell(v_m(x,t)) + \cdots \\ &= \sum_{i=0}^{\infty} \ell(v_i(x,t)), \end{aligned} \tag{7}$$

and

$$N(v(x,t)) = N(v_0(x,t)) + \sum_{i=1}^{\infty}\left[N\left(\sum_{k=0}^{i} v_k(x,t)\right) - N\left(\sum_{k=0}^{i-1} v_k(x,t)\right)\right], \tag{8}$$

substituting Eqs. (7) and (8) in Eq. (5) gives

$$\sum_{i=0}^{\infty} v_i(x,t) = \sum_{i=0}^{\infty} \ell(v_i(x,t)) + N(v_0(x,t)) + \sum_{i=1}^{\infty}\left[N\left(\sum_{k=0}^{i} v_k(x,t)\right) - N\left(\sum_{k=0}^{i-1} v_k(x,t)\right)\right] + h(x,t). \tag{9}$$

From Eq. (9), we have

$$\begin{aligned} v_0 &= v(x,0) = h(x), \quad \text{initial condition} \\ v_1 &= \ell(v_0) + N(v_0), \\ v_2 &= \ell(v_1) + N(v_0 + v_1) - N(v_0), \\ v_3 &= \ell(v_2) + N(v_0 + v_1 + v_2) - N(v_0 + v_1), \\ &\vdots \\ v_{n+1} &= \ell(v_n) + N\left(\sum_{i=0}^{n} v_i(x,t)\right) - N\left(\sum_{i=0}^{n-1} v_i(x,t)\right), \quad i = 1,\ 2, \ldots \end{aligned} \tag{10}$$

3. Description of the method

In this paper, we consider a non-linear parabolic equation of the form (1), that is,

$$D_t^{\alpha} v = v_{xx} + \gamma v + \psi v^{\tau}, \tag{11}$$

with initial condition

$$v(x,\ 0) = h(x). \tag{12}$$

Introducing J^{α} to both sides of Eq. (11) results in an integral equation of the form Eq. (5), that is,

$$v(x,\ t) = \ell(v(x,t)) + N(v(x,t)) + h(x,t), \tag{13}$$

where

$$\begin{aligned} v_0 &= h(x), \\ \ell(v(x,t)) &= J^{\alpha}\left(v_{xx} + v\right), \\ N(v(x,t)) &= J^{\alpha}\left(\psi v^{\tau}\right). \end{aligned} \tag{14}$$

We then solve for the series solution using Eq. (10).

Definition 1: Gamma function is defined as:

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du, \quad \text{where } \mathbb{R}(x) > 0. \tag{15}$$

Definition 2: Let $\alpha > 0$ and m be an integer such that $m - 1 < \alpha < m$, then the Riemann-Liouville fractional derivative of v is defined as [22, 23]:

$$D^{\alpha} v(x,t) = \begin{cases} J^{m-\alpha} v^{(m)}(x,t), & m - 1 < \alpha < m \\ v^{(m)}(x,t), & \alpha = m \end{cases}, \tag{16}$$

where

$$J^{\alpha} v(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} v(x,\sigma) d\sigma, \quad \alpha, t > 0. \tag{17}$$

Note that when $\alpha = 0$, then $J^{\alpha} v(x,t) = v(x,t)$. Other properties for function $v(x,t)$ are:

(a)

$$J^{\alpha}J^{\nu}v(x,t) = J^{\nu}J^{\alpha}v(x,t) = J^{\alpha+\nu}v(x,t). \tag{18}$$

(b)

$$J^{\alpha}t^{\xi} = \frac{\Gamma(\xi+1)}{\Gamma(\alpha+\xi+1)}t^{\xi+\alpha}, \quad \alpha,\nu \geq 0, \ \xi > -1. \tag{19}$$

Eq. (16) must satisfy the following properties:

$$D^{\alpha}\left[\rho v(x,t) + \delta u(x,t)\right] = \rho D^{\alpha}v(x,t) + \delta D^{\alpha}u(x,t), \quad \delta,\rho \in \mathbb{R}, \tag{20}$$

$$J^{\alpha}D^{\alpha}v(x,t) = v(x,t) - \sum_{i=0}^{m-1} v^{i}(x,0)\frac{t^{i}}{i!}, \qquad \text{and} \tag{21}$$

$$D^{\alpha}J^{\alpha}v(x,t) = v(x,t). \tag{22}$$

4. Numerical Implementation

In this section, we demonstrate the iterative method on some selected wave equations from the literature. For ease of comparison, we compute the absolute errors E_i at $t = T$ defined by

$$E_i = |v(x_i,T) - v_n(x_i,T)|, \tag{23}$$

and compare with the existing results in the literature.

Example 4.1. *Consider the Allen-Cahn equation [1, 24] with $\alpha = 1$:*

$$D_t^{\alpha}v = v_{xx} + v - v^3, \tag{24}$$

subject to initial condition:

$$v(x,0) = -\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{221}{625}x\right), \tag{25}$$

and the exact solution is

$$v(x,t) = -\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{221}{625}x - \frac{3}{4}t\right). \tag{26}$$

Applying J^{α} on both sides of Eq. (24), then Eqs. (24) and (25) become:

$$v(x,t) = \ell(v) + N(v) + h(x,t),$$

where

$$v_0 = h(x) = -\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{221}{625}x\right), \tag{27}$$

$$\ell(v) = J^{\alpha}\left(v_{xx} + v\right), \ \text{ and}$$

$$N(v) = J^{\alpha}(-v^3).$$

Applying Eq. (10), we have

$$\begin{aligned} v_1 &= \ell(v_0) + N(v_0) \\ &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[-\frac{103}{3125000}\tanh\left(\frac{221x}{625}\right) + \frac{3}{8}\tanh^2\left(\frac{221x}{625}\right) + \frac{103}{3125000}\tanh^3\left(\frac{221x}{625}\right) - \frac{3}{8}\right] \\ &= -\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{\text{sech}^2\left(\frac{221x}{625}\right)\left(1171875 + 103\tanh\left(\frac{221x}{625}\right)\right)}{3125000}\right], \end{aligned} \tag{28}$$

Table 1: Comparison of absolute errors for Example 4.1.

	$t = 0.001$		$t = 0.005$		$t = 0.009$		$t = 0.01$	
x	present	MVIA-II[1]	present	MVIA-II[1]	present	MVIA-II[1]	present	MVIA-II[1]
0.1	1.15×10^{-9}	1.62×10^{-5}	5.78×10^{-9}	8.10×10^{-5}	1.11×10^{-8}	1.45×10^{-4}	1.26×10^{-8}	1.62×10^{-4}
0.2	2.31×10^{-9}	1.61×10^{-5}	1.18×10^{-8}	8.07×10^{-5}	2.28×10^{-8}	1.45×10^{-4}	2.60×10^{-8}	1.62×10^{-4}
0.3	3.44×10^{-9}	1.60×10^{-5}	1.76×10^{-8}	8.02×10^{-5}	3.41×10^{-8}	1.43×10^{-4}	3.89×10^{-8}	1.61×10^{-4}
0.4	4.53×10^{-9}	1.59×10^{-5}	2.32×10^{-8}	7.96×10^{-5}	4.47×10^{-8}	1.42×10^{-4}	5.09×10^{-8}	1.59×10^{-4}
0.5	5.59×10^{-9}	1.57×10^{-5}	2.86×10^{-8}	7.87×10^{-5}	5.47×10^{-8}	1.40×10^{-4}	6.21×10^{-8}	1.58×10^{-4}
0.6	6.58×10^{-9}	1.55×10^{-5}	3.36×10^{-8}	7.76×10^{-5}	6.39×10^{-8}	1.38×10^{-4}	7.24×10^{-8}	1.56×10^{-4}
0.7	7.52×10^{-9}	1.53×10^{-5}	3.83×10^{-8}	7.64×10^{-5}	7.23×10^{-8}	1.35×10^{-4}	8.16×10^{-8}	1.53×10^{-4}
0.8	8.39×10^{-9}	1.50×10^{-5}	4.26×10^{-8}	7.50×10^{-5}	7.97×10^{-8}	1.33×10^{-4}	8.97×10^{-8}	1.50×10^{-4}
0.9	9.18×10^{-9}	1.47×10^{-5}	4.64×10^{-8}	7.54×10^{-5}	8.62×10^{-8}	1.30×10^{-4}	9.68×10^{-8}	1.44×10^{-4}

$$\begin{aligned} v_2 &= \ell(v_1) + N(v_0+v_1) - N(v_0) \\ &= J^\alpha\left[v_1 + (v_1)_{xx}\right] + J^\alpha\left[-(v_0+v_1)^3 + v_0^3\right] \\ &= \frac{-3t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\left[\left(-\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{221x}{625}\right)\right)\frac{\operatorname{sech}^4\left(\frac{221x}{625}\right)\left(1171875+103\tanh\left[\frac{221x}{625}\right]\right)^2}{9765625000000}\right] \\ &+ \frac{t^{4\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+4\alpha)}\left[\frac{\operatorname{sech}^6\left(\frac{221x}{625}\right)\left(1171875+103\tanh\left(\frac{221x}{625}\right)\right)^3}{30517578125000000000}\right] \\ &- \frac{t^{2\alpha}}{4882812500000\Gamma(1+2\alpha)}\operatorname{sech}^5\left(\frac{221x}{625}\right)\left[-241406250\cosh\left(\frac{221x}{625}\right)\right. \\ &\left.+120703125\cosh\left(\frac{663x}{625}\right) + 686524746338\sinh\left(\frac{221x}{625}\right) + 686645513117\sinh\left(\frac{663x}{625}\right)\right]. \end{aligned} \tag{29}$$

Substituting Eqs. (27), (28), and (29) in Eq. (6), we obtained a series solution in the form:

$$\begin{aligned} V_2 &= v_0 + v_1 + v_2 \\ &= -\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{221x}{625}\right) - \frac{t^\alpha\left(\frac{\operatorname{sech}^2\left(\frac{221x}{625}\right)\left(1171875+103\tanh\left(\frac{221x}{625}\right)\right)}{3125000}\right)}{\Gamma(1+\alpha)} + \frac{t^{4\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+4\alpha)}\frac{\operatorname{sech}^6\left(\frac{221x}{625}\right)\left(1171875+103\tanh\left(\frac{221x}{625}\right)\right)^3}{30517578125000000000} - \\ &\frac{3t^{3\alpha}\Gamma(1+2\alpha)\left(-1+\tanh\left[\frac{221x}{625}\right]\right)}{2\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\frac{\operatorname{sech}\left[\frac{221x}{625}\right]^4\left(1171875+103\tanh\left[\frac{221x}{625}\right]\right)^2}{9765625000000} \\ &- \frac{t^{2\alpha}\operatorname{sech}\left[\frac{221x}{625}\right]^5}{4882812500000\Gamma(1+2\alpha)}\left(-241406250\cosh\left[\frac{221x}{625}\right] + 120703125\cosh\left[\frac{663x}{625}\right] + 686524746338\sinh\left[\frac{221x}{625}\right]\right. \\ &\left.+686645513117\sinh\left[\frac{663x}{625}\right]\right). \end{aligned} \tag{30}$$

More accurate results can be obtained by finding $V_m,\ m = 3,\ 4,\ 5,\cdots$.

Table 1 is the absolute errors corresponding to Eq. (30) and corresponding figure is Figure 2, Figure 1 is the behaviour of exact and its corresponding approximate solution for Example 4.1.

Example 4.2. *Consider*

$$D_t^\alpha v = v_{xx} + v - v^3, \quad with\ \ \alpha = 1, \tag{31}$$

subject to initial condition:

$$v(x,0) = \left(1+\exp\left(-\frac{1}{\sqrt{2}}x\right)\right)^{-1}, \tag{32}$$

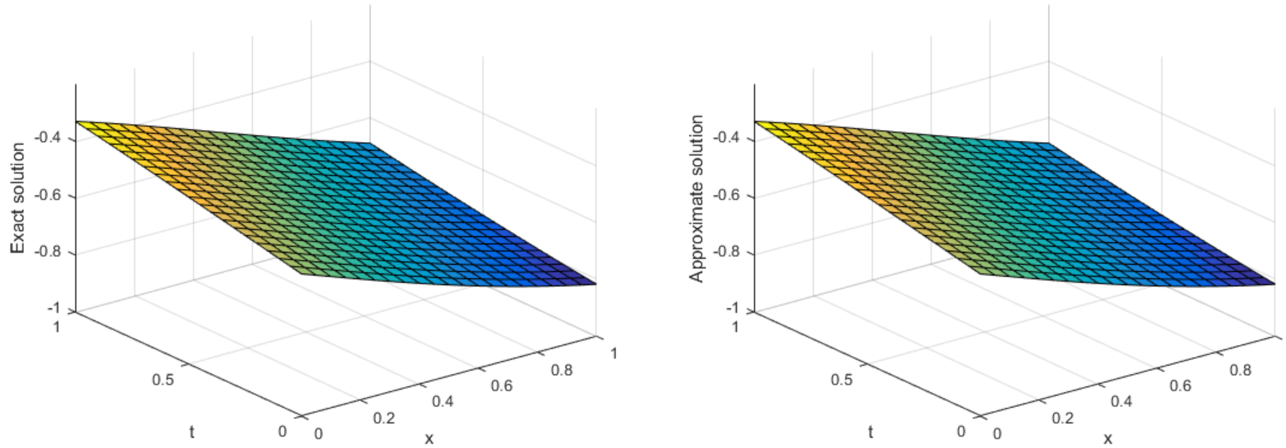

Figure 1: Exact solution and its corresponding approximate solution for Example 4.1.

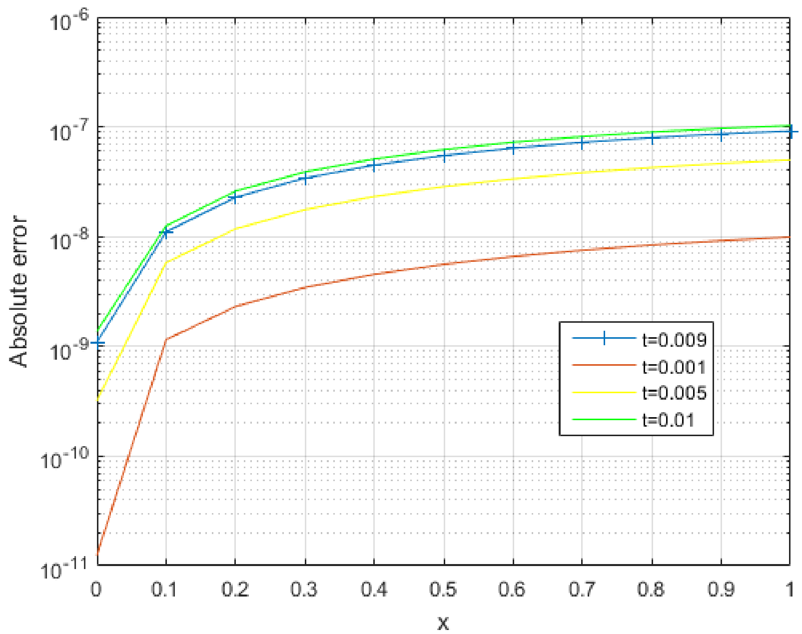

Figure 2: Absolute errors with different values of t.

and the exact solution is

$$v(x,t) = \left(1 + \exp\left(-\frac{1}{\sqrt{2}}x - \frac{3}{2}t\right)\right)^{-1}. \tag{33}$$

Table 2 is the absolute errors and their corresponding results from the literature, while Figure 2 is the graph of the results obtained in Table 2.

Example 4.3. *Consider*

$$D_t^{\alpha} v = v_{xx} + v - v^2, \tag{34}$$

when $\alpha = 1$. *Eq. (34) becomes Newell-Whitehead equation [25], having initial condition:*

$$v(x,0) = \left(1 + \exp\left(\frac{3}{\sqrt{10}}x\right)\right)^{-\frac{2}{3}}, \tag{35}$$

Table 2: Comparison of absolute errors for Example 4.2.

	$t = 0.001$		$t = 0.005$		$t = 0.009$		$t = 0.01$	
x	present	MVIA-II[1]	present	MVIA-II[1]	present	MVIA-II[1]	present	MVIA-II[1]
0.1	9.93×10^{-9}	3.77×10^{-5}	2.49×10^{-7}	2.88×10^{-4}	8.06×10^{-7}	5.19×10^{-4}	9.96×10^{-7}	5.76×10^{-4}
0.2	1.98×10^{-8}	5.74×10^{-5}	4.95×10^{-7}	2.87×10^{-4}	1.60×10^{-6}	5.17×10^{-4}	1.98×10^{-6}	5.74×10^{-4}
0.3	2.94×10^{-8}	5.71×10^{-5}	7.36×10^{-7}	2.85×10^{-4}	2.39×10^{-6}	5.13×10^{-4}	2.95×10^{-6}	5.70×10^{-4}
0.4	3.87×10^{-8}	5.66×10^{-5}	9.70×10^{-7}	2.83×10^{-4}	3.15×10^{-6}	5.08×10^{-4}	3.89×10^{-6}	5.65×10^{-4}
0.5	4.77×10^{-8}	5.60×10^{-5}	1.19×10^{-6}	2.80×10^{-4}	3.88×10^{-6}	5.03×10^{-4}	4.79×10^{-6}	5.58×10^{-4}
0.6	5.62×10^{-8}	5.52×10^{-5}	1.41×10^{-6}	2.76×10^{-4}	4.57×10^{-6}	4.96×10^{-4}	5.64×10^{-6}	5.51×10^{-4}
0.7	6.42×10^{-8}	5.43×10^{-5}	1.61×10^{-6}	2.71×10^{-4}	5.22×10^{-6}	4.88×10^{-4}	6.44×10^{-6}	5.42×10^{-4}
0.8	7.16×10^{-8}	5.33×10^{-5}	1.79×10^{-6}	2.66×10^{-4}	5.82×10^{-6}	4.79×10^{-4}	7.19×10^{-6}	5.31×10^{-4}
0.9	7.84×10^{-8}	5.22×10^{-5}	1.96×10^{-6}	2.61×10^{-4}	6.37×10^{-6}	4.69×10^{-4}	7.87×10^{-6}	5.20×10^{-4}

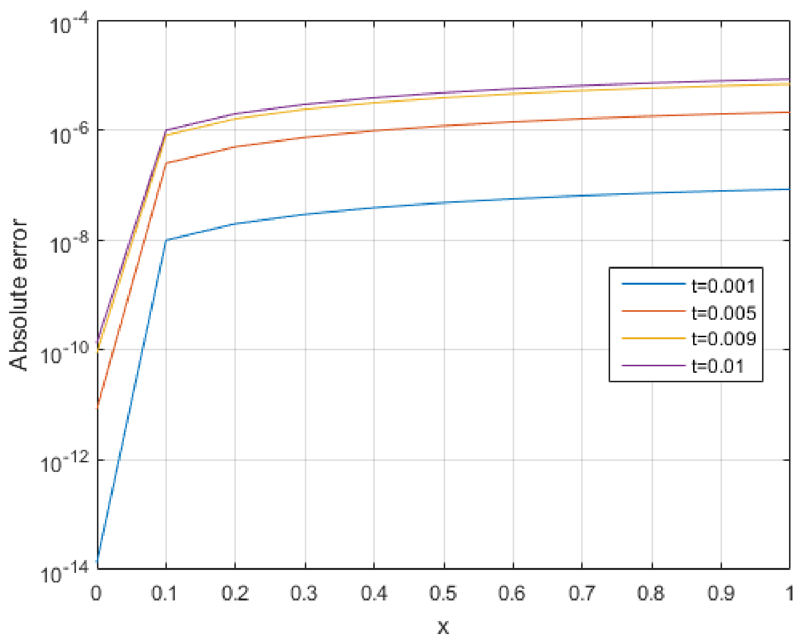

Figure 3: Absolute errors with different values of t for Example 4.2.

and the exact solution is

$$v(x,t) = \left[\frac{1}{2}\left\{1 + \tanh\left(-\frac{3}{2\sqrt{10}}\left(x - \frac{7}{\sqrt{10}}t\right)\right)\right\}\right]^{\frac{2}{3}}. \tag{36}$$

Table 3 is the absolute errors as compared with the results obtained in the literature; and Table 4 is the absolute errors as the values of t get flatter.

Example 4.4. *Consider Eq. (34) with $\alpha = 1$ subject to the initial condition:*

$$v(x,0) = \left(1 + \exp\left(-\frac{1}{\sqrt{6}}x\right)\right)^{-2}, \tag{37}$$

and the exact solution is

$$v(x,t) = \left(1 + \exp\left(\frac{1}{\sqrt{6}}x - \frac{5}{6}t\right)\right)^{-2}. \tag{38}$$

Table 3: Comparison of absolute errors for Example 4.3.

	$t = 0.1$		$t = 0.3$		$t = 0.5$	
x	present	MVIA-II[1]	present	MVIA-II[1]	present	MVIA-II[1]
0.1	1.15×10^{-5}	9.26×10^{-6}	1.12×10^{-4}	3.86×10^{-5}	1.61×10^{-3}	1.62×10^{-3}
0.2	1.2×10^{-5}	1.14×10^{-5}	2.75×10^{-4}	1.52×10^{-4}	2.68×10^{-3}	1.87×10^{-4}
0.4	4.90×10^{-6}	1.27×10^{-5}	2.17×10^{-4}	2.92×10^{-4}	1.32×10^{-3}	1.67×10^{-3}
0.6	1.30×10^{-5}	1.32×10^{-5}	3.06×10^{-4}	3.63×10^{-4}	1.01×10^{-4}	2.61×10^{-3}
0.8	1.30×10^{-5}	1.28×10^{-5}	3.60×10^{-4}	3.65×10^{-4}	2.20×10^{-3}	2.93×10^{-3}
1.0	1.31×10^{-5}	1.18×10^{-5}	3.62×10^{-4}	3.13×10^{-4}	2.63×10^{-3}	2.71×10^{-3}

Table 4: Absolute errors for Example 4.3 with nearly flat t.

x	$t = 0.001$	$t = 0.005$	$t = 0.009$	$t = 0.01$
0	2.84×10^{-11}	3.59×10^{-9}	2.12×10^{-8}	2.92×10^{-8}
0.2	7.39×10^{-12}	9.58×10^{-10}	5.78×10^{-9}	8.00×10^{-9}
0.4	9.55×10^{-12}	1.17×10^{-9}	6.72×10^{-9}	9.17×10^{-9}
0.6	2.09×10^{-11}	2.61×10^{-9}	1.51×10^{-8}	2.08×10^{-8}
0.8	2.61×10^{-11}	3.27×10^{-9}	1.91×10^{-8}	2.62×10^{-8}
1.0	2.56×10^{-11}	3.21×10^{-9}	1.88×10^{-8}	2.58×10^{-8}

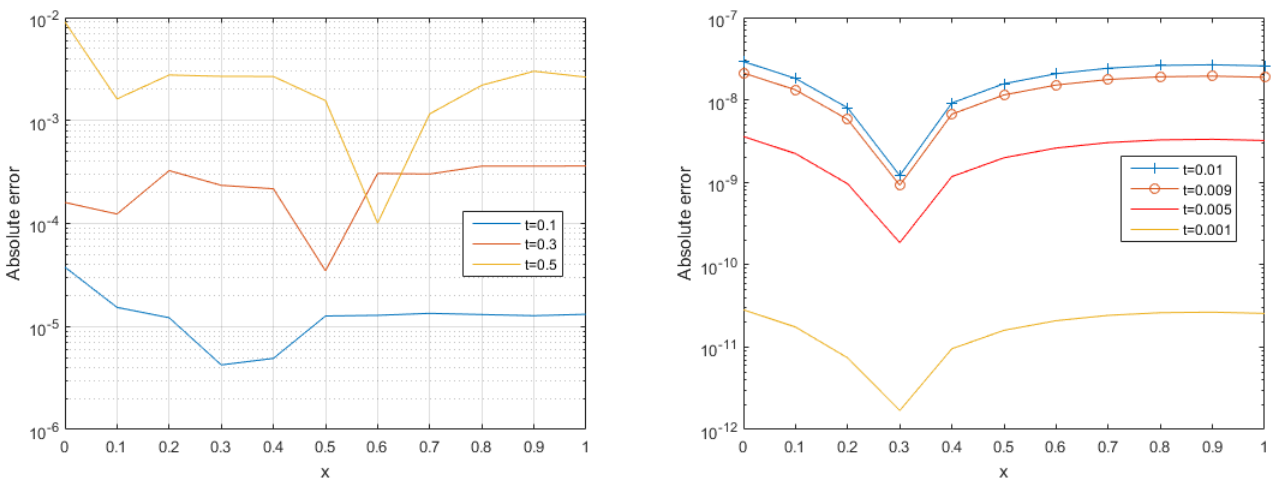

Figure 4: Comparison of the absolute errors at different values of t for Example 4.3.

Table 5 is the comparison of the absolute errors with the results obtained in the literature; and Table 6 is the absolute errors as the values of t get flatter.

Example 4.5. *Consider*

$$D_t^{\alpha} v = v_{xx} + 6v - 6v^2, \tag{39}$$

with $\alpha = 1$, *we have Fisher equation [26], having initial condition:*

$$v(x, 0) = [1 + \exp(x)]^{-2}, \tag{40}$$

and the exact solution is

$$v(x, t) = [1 + \exp(x - 5t)]^{-2}. \tag{41}$$

Table 5: Absolute errors for Example 4.4.

t	x	MVIA-II[1]	VIM [27]	present
0	1	2.50×10^{-16}	0	2.78×10^{-17}
0.2	1	5.6×10^{-7}	2.1×10^{-6}	4.8×10^{-5}
0.4	1	8.3×10^{-7}	5.3×10^{-5}	3.1×10^{-4}
0.6	1	1.4×10^{-5}	3.2×10^{-4}	8.2×10^{-4}
0.8	1	5.8×10^{-5}	1.1×10^{-2}	9.1×10^{-4}
1.0	1	1.6×10^{-4}	2.6×10^{-2}	9.5×10^{-4}

Table 6: Absolute errors for Example 4.4 with nearly flat t.

x	$t = 0.001$	$t = 0.005$	$t = 0.009$	$t = 0.01$
0	2.41×10^{-12}	2.98×10^{-10}	1.72×10^{-9}	2.36×10^{-9}
0.2	3.24×10^{-12}	4.03×10^{-10}	2.33×10^{-9}	3.20×10^{-9}
0.4	4.14×10^{-12}	5.18×10^{-10}	3.00×10^{-9}	4.12×10^{-9}
0.6	5.13×10^{-12}	6.39×10^{-10}	3.71×10^{-9}	5.08×10^{-9}
0.8	6.12×10^{-12}	7.62×10^{-10}	4.42×10^{-9}	6.07×10^{-9}
1.0	7.08×10^{-12}	8.82×10^{-10}	5.13×10^{-9}	7.03×10^{-9}

Table 7: Absolute errors for Example 4.5 with $t = 0.4$.

x	MVIA-II[1]	ADM [26]	MVIM [26]	present
0	3.95×10^{-2}	5.75×10^{-2}	5.02×10^{-2}	2.42×10^{-2}
0.2	3.01×10^{-2}	1.62×10^{-1}	5.27×10^{-2}	4.24×10^{-2}
0.4	1.47×10^{-2}	1.39×10^{-1}	4.12×10^{-2}	4.25×10^{-2}
0.6	6.89×10^{-3}	1.52×10^{-1}	2.25×10^{-2}	2.60×10^{-2}
0.8	1.74×10^{-2}	1.44×10^{-1}	5.29×10^{-3}	1.72×10^{-3}
1.0	1.81×10^{-2}	1.19×10^{-1}	4.24×10^{-3}	3.40×10^{-2}

Table 8: Absolute errors for Example 4.5 with nearly flat t.

x	$t = 0.001$	$t = 0.005$	$t = 0.009$	$t = 0.01$
0	5.14×10^{-10}	6.11×10^{-8}	3.37×10^{-7}	4.56×10^{-7}
0.2	9.87×10^{-10}	1.20×10^{-7}	6.77×10^{-7}	9.22×10^{-7}
0.4	1.50×10^{-9}	1.84×10^{-7}	1.06×10^{-6}	1.44×10^{-6}
0.6	1.96×10^{-9}	2.42×10^{-7}	1.39×10^{-6}	1.91×10^{-6}
0.8	2.27×10^{-9}	2.81×10^{-7}	1.63×10^{-6}	2.23×10^{-6}
1.0	2.39×10^{-9}	2.98×10^{-7}	1.73×10^{-6}	2.38×10^{-6}

Table 7 shows the comparison of the absolute errors relative to the results from the literature; and Table 8 shows the absolute errors as the values of t get flatter.

5. Discussion of results and conclusion

5.1. Discussion of results

Tables 1-3, 5 and 7 are the absolute errors obtained from Examples 4.1-4.5, respectively, and their corresponding results from the literature, while Tables 4, 6 and 8 are the absolute errors obtained when the values of t are getting flatter (that's a very small value of t). Figures 1 is the exact and approximate solutions for Example 4.1, while Figures 2 and 3 shows the relationship between the absolute errors at different values of t for Examples 4.1 and 4.2 respectively. Figure 4 shows the comparison of the absolute errors at different values of t, for Example 4.3.

5.2. Conclusion

In this paper, we have studied and analysed a new iterative method for solving time-fractional non-linear equations. The Examples considered are Allen-Cahn, Newell-Whitehead, and Fisher equations by putting $\alpha = 1$. We obtained an approximate solution for each example, and computational results were obtained and tabulated. Obviously, from the tables of results, the proposed method is effective and accurate for solving Allen-Cahn, Newell-Whitehead, and Fisher equations. All computations were done using the Mathematica package, and the graphs were plotted using the MATLAB package. The scheme is also applicable to fractional form by changing the value of α in the equations Section 3 to enhance further study.

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