Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://asr.nsps.org.ng/index.php/asr

African Scientific Reports 3 (2024) 156

African Scientific Reports

A fifth order block methods for solving second-order stiff ordinary differential equations using trigonometric functions and polynomial function as the basis function

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Abstract

The numerical solution of second-order ordinary differential equations (ODEs) is examined in this work through a four-step linear multistep method. It employs a combination of trigonometric and polynomial functions as the approximate solution to the general second-order ordinary differential equations (ODEs). The method was developed using interpolation and collocation techniques. This methodology involves interpolating the basis function at two points and subsequently collocating it across all points, ensuring a robust scheme. To assess its efficacy, we solved three initial value problems (IVPs) associated with stiff differential equations. Through this examination, we established the method's core characteristics: consistency, zero stability, and consequently convergence. This thorough analysis demonstrates its reliability and suitability for resolving second-order ordinary differential equations. The comparison of our newly derived block method against existing approaches reveals its superiority. Our method's performance, evaluated across a spectrum of stiff second-order ordinary differential equations, surpasses the outcomes obtained from established authors. This substantiates its efficiency and effectiveness in addressing these mathematical challenges. This study marks a significant advancement by introducing a robust approach that not only accurately solves second-order ordinary differential equations but also streamlines the computational process. By integrating trigonometric and polynomial functions and leveraging interpolation and collocation techniques, our method stands out for its accuracy, stability, and convergence properties, offering a promising avenue for future research in this domain.

DOI:10.46481/asr.2024.3.2.156

Keywords: Trigonometric function, Polynomial function, Collocation, Interpolation, Stiff ODEs, Block method

Article History:
Received: 08 November 2023
Received in revised form: 29 February 2024
Accepted for publication: 29 May 2024
Published: 29 June 2024

1. Introduction

Presented in this paper is the numerical solution of general second-order ordinary differential equations initial value problems (IVP) of the form:

$$y'' = f(x, y, y'), \quad y(a) = \eta_0, \; y'(a) = \eta_1, \quad x \in [a, b]. \tag{1}$$

These second-order ODEs, which are used to solve numerous problems in physical sciences and engineering, are used to model a variety of natural processes. Many forms of the equation (1) may be difficult to solve analytically. As a result, numerical systems

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are frequently developed to solve them. Because a lot of problems of the form (1) are difficult to solve, approximate numerical integrations are commonly utilized to solve them. These second-order ODEs are frequently transformed to similar first-order ordinary differential equations and solved using the appropriate method. Among others who have discussed the reduction technique are Refs. [1–3].

To circumvent the problem of changing equation (1) to an equal system of first order ODEs, scholars like Awoyemi & Olanegan *et al.* [3, 4] proposed a linear multistep technique for solving equation (1) directly. According to Ref. [3], the continuous linear multistep method outperforms the discrete method in terms of error estimation by providing a simplified coefficient for more analytical work at various points and ensuring easy appropriation of solution at all interior points of the integration interval.

Authors such as Omar & Kuboye [6] and Onumanyi *et al.* [7] have presented continuous linear multistep techniques. These scholars used the predictor-corrector and Taylor series expansion to acquire starting values for their approaches. The predictor-corrector approach is expensive, according to Ref. [8], since subroutines are difficult to construct due to the specific methods necessary to offer beginning values and adjust the step size, resulting in longer computer time and more human effort. The correctors are not in the same order as the predictors. As a result, the accuracy of the method is very poor.

To rise above the drawbacks of the reduction and predictor-corrector methods, researchers such as Refs. [8–10] developed the block technique. The aim of this paper is to develop and implement a new continuous four step linear multistep block method that is zero-stable, consistent, and convergent for direct and accurate solution of second-order ODEs of initial value problems using trigonometric and polynomial functions as the basis function.

2. Derivation of the scheme

In deriving this method, basis function of the form,

$$y(x) = \sum_{n=0}^{4} a_n x^n + a_5 \sin(\omega x) + a_6 \cos(\omega x), \tag{2}$$

is considered as the approximate solution to equation (1). The second derivative of equation (2) is obtained as:

$$y^{''}(x) = \sum_{n=2}^{4} n(n-1) a_n x^{n-2} - a_5(\omega^2)\sin(\omega x) - a_6(\omega^2)\cos(\omega x). \tag{3}$$

Collocating equation (3) at $x = x_{n+j},\ j = 0(1)4$ and interpolating equation (2) at $x = x_{n+j},\ j = 0, 1$. These equations are then combined together to give a non linear system of equations of the form:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \sin wx_n & \cos wx_n \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \sin wx_{n+1} & \cos wx_{n+1} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & -w^2 \sin wx_n & -w^2 \cos wx_n \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & -w^2 \sin wx_{n+1} & -w^2 \cos wx_{n+1} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & -w^2 \sin wx_{n+2} & -w^2 \cos wx_{n+2} \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & -w^2 \sin wx_{n+3} & -w^2 \cos wx_{n+3} \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & -w^2 \sin wx_{n+4} & -w^2 \cos wx_{n+4} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}. \tag{4}$$

Gaussian elimination technique is used in finding the values of $a_n^{'}s$ in equation (4) which are then substituted into equation (2) to produce a continuous implicit method of the form

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} \right], \tag{5}$$

where the coefficients of y_{n+j} and f_{n+j} gives

$$
\begin{aligned}
\alpha_0 &= -1,\\
\alpha_1 &= 2,\\
\beta_0 &= \frac{14\sin(wh)h^2w^2 - 13h^2w^2\sin(2wh) - 36\sin(wh) + 36\sin(2wh) - 12\sin(3wh)}{60\sin(wh)w^2 + 12w^2\sin(3wh) - 48w^2\sin(2wh)},\\
\beta_1 &= \frac{33\sin(wh)h^2w^2 - 12h^2w^2\sin(2wh) + 13w^2h^2\sin(3wh) + 84\sin(wh) - 96\sin(2wh) + 36\sin(3wh)}{60\sin(wh)w^2 + 12w^2\sin(3wh) - 48w^2\sin(2wh)},\\
\beta_2 &= \frac{\sin(wh)h^2w^2 - w^2h^2\sin(3wh) - 17h^2w^2\sin(2wh) - 18\sin(3wh) - 18\sin(wh) + 36\sin(2wh)}{30\sin(wh)w^2 - 24w^2\sin(2wh) + 6w^2\sin(3wh)},\\
\beta_3 &= \frac{21\sin(wh)h^2w^2 + 12h^2w^2\sin(2wh) + w^2h^2\sin(3wh) - 36\sin(wh) + 12\sin(3wh)}{60\sin(wh)w^2 + 12w^2\sin(3wh) - 48w^2\sin(2wh)},\\
\beta_4 &= \frac{-10\sin(wh)h^2w^2 - h^2w^2\sin(2wh) + 24\sin(wh) - 12\sin(2wh)}{20\sin(wh)w^2 + 4w^2 sin(3wh) - 16w^2\sin(2wh)}.
\end{aligned} \tag{6}
$$

Letting $u = \omega h$, the converted coefficients of the method in series form is given as

$$
\begin{aligned}
\beta_0 &= \frac{19}{240}h^2 + \frac{221}{60480}h^2u^2 + \frac{233}{1814400}h^2u^4 + \frac{199}{53222400}h^2u^6 + O(u^8),\\
\beta_1 &= \frac{17}{20}h^2 - \frac{79}{7560}h^2u^2 - \frac{79}{226800}h^2u^4 - \frac{61}{6652800}h^2u^6 + O(u^8),\\
\beta_2 &= \frac{7}{120}h^2 + \frac{19}{2016}h^2u^2 + \frac{83}{302400}h^2u^4 + \frac{1}{197120}h^2u^6 + O(u^8),\\
\beta_3 &= \frac{1}{60}h^2 - \frac{2}{945}h^2u^2 - \frac{1}{56700}h^2u^4 + \frac{1}{415800}h^2u^6 + O(u^8),\\
\beta_4 &= -\frac{1}{240}h^2 - \frac{31}{60480}h^2u^2 - \frac{67}{1814400}h^2u^4 - \frac{109}{53222400}h^2u^6 + O(u^8),
\end{aligned} \tag{7}
$$

or y_{n+3},

$$
\begin{aligned}
\alpha_0 &= -2,\\
\alpha_1 &= 3,\\
\beta_0 &= \frac{9h^2w^2\sin(2wh) - 6\sin(wh)h^2w^2 - 20\sin(2wh) + 16\sin(wh) + 8\sin(3wh)}{16w^2\sin(2wh) - 20\sin(wh)w^2 - 4w^2\sin(3wh)},\\
\beta_1 &= \frac{8h^2w^2\sin(2wh) - 37\sin(wh)h^2w^2 - 9w^2h^2\sin(3wh) + 48\sin(2wh) - 24\sin(wh) - 24\sin(3wh)}{16w^2\sin(2wh) - 20\sin(wh)w^2 - 4w^2\sin(3wh)},\\
\beta_2 &= \frac{\sin(wh)h^2w^2 - w^2h^2\sin(3wh) + 19h^2w^2\sin(2wh) - 12\sin(2wh) + 12\sin(3wh) - 12\sin(wh)}{8w^2\sin(2wh) - 10\sin(wh)w^2 - 2w^2\sin(3wh)},\\
\beta_3 &= \frac{29\sin(wh)h^2w^2 + w^2h^2\sin(3wh) + 8h^2w^2\sin(2wh) - 56\sin(wh) + 8\sin(3wh) + 16\sin(2wh)}{20\sin(wh)w^2 + 4w^2\sin(3wh) - 16w^2\sin(2wh)},\\
\beta_4 &= \frac{-10\sin(wh)h^2w^2 - h^2w^2\sin(2wh) + 24\sin(wh) - 12\sin(2wh)}{60\sin(wh)w^2 + 12w^2\sin(3wh) - 48w^2\sin(2wh)}.
\end{aligned} \tag{8}
$$

The converted coefficients of the method in series form is given as

$$
\begin{aligned}
\beta_0 &= \frac{37}{240}h^2 + \frac{137}{20160}h^2u^2 + \frac{19}{86400}h^2u^4 + \frac{289}{53222400}h^2u^6 + O(u^8),\\
\beta_1 &= \frac{9}{5}h^2 - \frac{-19}{1008}h^2u^2 - \frac{83}{151200}h^2u^4 - \frac{1}{98560}h^2u^6 + O(u^8),\\
\beta_2 &= \frac{37}{40}h^2 + \frac{53}{3360}h^2u^2 + \frac{11}{33600}h^2u^4 - \frac{19}{8870400}h^2u^6 + O(u^8),\\
\beta_3 &= \frac{2}{15}h^2 - \frac{11}{5040}h^2u^2 + \frac{17}{151200}h^2u^4 + \frac{173}{13305600}h^2u^6 + O(u^8),\\
\beta_4 &= -\frac{1}{80}h^2 - \frac{31}{20160}h^2u^2 - \frac{67}{604800}h^2u^4 - \frac{109}{17740800}h^2u^6 + O(u^8),
\end{aligned} \tag{9}
$$

for y_{n+4},

$$\begin{aligned}
\alpha_0 &= -3,\\
\alpha_1 &= 4,\\
\beta_0 &= \frac{2\sin(wh)h^2w^2 - 7h^2w^2\sin(2wh) - 6\sin(wh) - 6\sin(3wh) + 12\sin(2wh)}{10\sin(wh)w^2 + 2w^2\sin(3wh) - 8w^2\sin(2wh)},\\
\beta_1 &= \frac{35\sin(wh)h^2w^2 + 7w^2h^2\sin(3wh) - 4h^2w^2\sin(2wh) + 4\sin(wh) + 20\sin(3wh) - 32\sin(2wh)}{10\sin(wh)w^2 + 2w^2\sin(3wh) - 8w^2\sin(2wh)},\\
\beta_2 &= \frac{w^2h^2\sin(3wh) - 19h^2w^2\sin(2wh) - \sin(wh)h^2w^2 - 12\sin(3wh) + 12\sin(2wh) + 12\sin(wh)}{w^2\sin(3wh) + 5\sin(wh)w^2 - 4w^2\sin(2wh)},\\
\beta_3 &= \frac{-4h^2w^2\sin(2wh) - 31\sin(wh)h^2w^2 - 3w^2h^2\sin(3wh) + 36\sin(wh) - 12\sin(3wh)}{8w^2\sin(2wh) - 10\sin(wh)w^2 - 2w^2\sin(3wh)},\\
\beta_4 &= \frac{3h^2w^2\sin(2wh) + 6\sin(wh)h^2w^2 + 4\sin(2wh) - 14\sin(wh) + 2\sin(3wh)}{8w^2\sin(2wh) - 10\sin(wh)w^2 - 2w^2\sin(3wh)}.
\end{aligned} \tag{10}$$

The converted coefficients of the method in series form is given as

$$\begin{aligned}
\beta_0 &= \frac{9}{40}h^2 + \frac{19}{2016}h^2u^2 + \frac{83}{302400}h^2u^4 + \frac{1}{197120}h^2u^6 + O(u^8),\\
\beta_1 &= \frac{83}{30}h^2 - \frac{37}{1260}h^2u^2 - \frac{29}{37800}h^2u^4 - \frac{29}{3326400}h^2u^6 + O(u^8),\\
\beta_2 &= \frac{37}{20}h^2 + \frac{53}{1680}h^2u^2 + \frac{11}{16800}h^2u^4 - \frac{19}{4435200}h^2u^6 + O(u^8),\\
\beta_3 &= \frac{11}{10}h^2 - \frac{4}{315}h^2u^2 - \frac{1}{9450}h^2u^4 + \frac{1}{69300}h^2u^6 + O(u^8),\\
\beta_4 &= \frac{7}{120}h^2 + \frac{11}{10080}h^2u^2 - \frac{17}{302400}h^2u^4 - \frac{173}{26611200}h^2u^6 + O(u^8).
\end{aligned} \tag{11}$$

Equation (2) was differentiated and evaluated at all points which gives the following coefficients and its series form. Differentiating and evaluating at $x = x_n$ gives y'_n.

$$\begin{aligned}
\widehat{\alpha_0} &= -\frac{1}{h},\\
\widehat{\alpha_1} &= \frac{1}{h},\\
\widehat{\beta_0} &= \frac{-9h^2w^2\sin(2wh) + 14\sin(wh)h^2w^2 - 8wh\cos(3wh)}{32hw^2\sin(2wh) - 8hw^2\sin(3wh) - 40hw^2\sin(wh)}\\
&\quad + \frac{8wh\cos(2wh) - 16\sin(2wh) + 8\sin(3wh) + 8\sin(wh)}{32hw^2\sin(2wh) - 8hw^2\sin(3wh) - 40hw^2\sin(wh)},\\
\widehat{\beta_1} &= \frac{-20h^2w^2\sin(2wh) + 27w^2h^2\sin(3wh) + 7\sin(wh)h^2w^2}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{72wh\cos(3wh) - 24\cos(wh)hw - 48wh\cos(2wh) + 120\sin(2wh) - 72\sin(3wh) - 24\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{72wh\cos(3wh) - 24\cos(wh)hw - 48wh\cos(2wh) + 120\sin(2wh) - 72\sin(3wh) - 24\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_2} &= \frac{-11w^2h^2\sin(3wh) + 11\sin(wh)h^2w^2 - 7h^2w^2\sin(2wh) - 36wh\cos(3wh)}{48hw^2\sin(2wh) - 12hw^2\sin(3wh) - 60hw^2\sin(wh)}\\
&\quad + \frac{36\cos(wh)hw + 36\sin(3wh) - 36\sin(wh) - 36\sin(2wh)}{48hw^2\sin(2wh) - 12hw^2\sin(3wh) - 60hw^2\sin(wh)},\\
\widehat{\beta_3} &= \frac{-7w^2h^2\sin(3wh) - 20h^2w^2\sin(2wh) + 13\sin(wh)h^2w^2 - 48wh\cos(2wh)}{24hw^2\sin(3wh) + 120hw^2\sin(wh) - 96hw^2\sin(2wh)},\\
&\quad + \frac{72\cos(wh)hw - 24wh\cos(3wh) + 24\sin(3wh) + 24\sin(2wh) - 120\sin(wh)}{24hw^2\sin(3wh) + 120hw^2\sin(wh) - 96hw^2\sin(2wh)},\\
\widehat{\beta_4} &= \frac{7h^2w^2\sin(2wh) - 2\sin(wh)h^2w^2 + 24wh\cos(2wh) - 24\cos(wh)hw - 24\sin(2wh) + 48\sin(wh)}{24hw^2\sin(3wh) + 120hw^2\sin(wh) - 96hw^2\sin(2wh)}.
\end{aligned} \tag{12}$$

The converted coefficients of the method in series form is given as

$$\begin{aligned}
\widehat{\beta_0} &= -\frac{367}{1440}h - \frac{199}{24192}hu^2 - \frac{1543}{3628800}hu^4 - \frac{281}{9123840}hu^6 + O(u^8),\\
\widehat{\beta_1} &= -\frac{3}{8}h + \frac{337}{15120}hu^2 + \frac{457}{453600}hu^4 - \frac{289}{4435200}hu^6 + O(u^8),\\
\widehat{\beta_2} &= \frac{47}{240}h - \frac{353}{20160}hu^2 - \frac{19}{40320}hu^4 - \frac{569}{53222400}hu^6 + O(u^8),\\
\widehat{\beta_3} &= -\frac{29}{360}h + \frac{1}{945}hu^2 - \frac{43}{113400}hu^4 - \frac{127}{2494800}hu^6 + O(u^8),\\
\widehat{\beta_4} &= \frac{7}{480}h + \frac{289}{120960}hu^2 + \frac{139}{518400}hu^4 + \frac{2899}{106444800}h^2u^6 + O(u^8).
\end{aligned} \tag{13}$$

Differentiating and evaluating at $x = x_n$ gives y'_{n+1}.

$$\begin{aligned}
\widehat{\alpha_0} &= -\frac{1}{h},\\
\widehat{\alpha_1} &= \frac{1}{h},\\
\widehat{\beta_0} &= \frac{26\sin(wh)h^2w^2 - 19h^2w^2\sin(2wh) - 24\cos(wh)hw}{120hw^2\sin(wh) - 96hw^2\sin(2wh) + 24hw^2\sin(3wh)}\\
&+ \frac{24wh\cos(2wh) - 24\sin(wh) + 48\sin(2wh) - 24\sin(3wh)}{120hw^2\sin(wh) - 96hw^2\sin(2wh) + 24hw^2\sin(3wh)},\\
\widehat{\beta_1} &= \frac{23\sin(wh)h^2w^2 - 16h^2w^2\sin(2wh) + 19w^2h^2\sin(3wh) + 48\cos(wh)hw - 72wh\cos(2wh)}{120hw^2\sin(wh) - 96hw^2\sin(2wh) + 24hw^2\sin(3wh)}\\
&+ \frac{24wh + 24\sin(wh) - 120\sin(2wh) + 72\sin(3wh)}{120hw^2\sin(wh) - 96hw^2\sin(2wh) + 24hw^2\sin(3wh)},\\
\widehat{\beta_2} &= \frac{5\sin(wh)h^2w^2 - 5w^2h^2\sin(3wh) - 13h^2w^2\sin(2wh)}{60hw^2\sin(wh) - 48hw^2\sin(2wh) + 12hw^2\sin(3wh)}\\
&+ \frac{36wh\cos(2wh) - 36wh - 36\sin(3wh) + 36\sin(wh) + 36\sin(2wh)}{60hw^2\sin(wh) - 48hw^2\sin(2wh) + 12hw^2\sin(3wh)},\\
\widehat{\beta_3} &= \frac{-16h^2w^2\sin(2wh) - 3w^2h^2\sin(3wh) - 7\sin(wh)h^2w^2}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&+ \frac{24wh\cos(2wh) + 48\cos(wh)hw - 72wh - 24\sin(2wh) - 24\sin(3wh) + 120\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_4} &= \frac{h^2w^2\sin(2wh) + 2\sin(wh)h^2w^2 - 8\cos(wh)hw + 8wh + 8\sin(2wh) - 16\sin(wh)}{32hw^2\sin(2wh) - 8hw^2\sin(3wh) - 40hw^2\sin(wh)}.
\end{aligned} \tag{14}$$

The converted coefficients of the method in series form is given as

$$\begin{aligned}
\widehat{\beta_0} &= \frac{3}{32}h + \frac{731}{120960}hu^2 + \frac{1439}{3628800}hu^4 - \frac{157}{5068800}hu^6 + O(u^8),\\
\widehat{\beta_1} &= \frac{47}{90}h - \frac{97}{6048}hu^2 - \frac{853}{907200}hu^4 - \frac{5317}{79833600}hu^6 + O(u^8),\\
\widehat{\beta_2} &= -\frac{41}{240}h + \frac{(239}{20160}hu^2 + \frac{89}{201600}hu^4 + \frac{743}{53222400}hu^6 + O(u^8),\\
\widehat{\beta_3} &= \frac{1}{15}h + \frac{1}{4320}hu^2 + \frac{319}{907200}hu^4 + \frac{1277}{26611200}hu^6 + O(u^8),\\
\widehat{\beta_4} &= \frac{17}{1440}h - \frac{253}{120960}hu^2 - \frac{181}{725760}hu^4 - \frac{1681}{63866880}h^2u^6 + O(u^8).
\end{aligned} \tag{15}$$

Differentiating and evaluating at $x = x_n$ gives y'_{n+2}.

$$
\begin{aligned}
\widehat{\alpha_0} &= -\frac{1}{h},\\
\widehat{\alpha_1} &= \frac{1}{h},\\
\widehat{\beta_0} &= \frac{29h^2w^2\sin(2wh) - 22\sin(wh)h^2w^2 - 24wh\cos(wh) + 24wh - 48\sin(2wh) + 24\sin(3wh) + 24\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_1} &= \frac{29w^2h^2\sin(3wh) + 113\sin(wh)h^2w^2 - 28h^2w^2\sin(2wh) - 48wh\cos(wh) + 48wh + 72\sin(3wh) + 24\sin(wh) - 120\sin(2wh)}{24hw^2\sin(3wh) + 120hw^2\sin(wh) - 96hw^2\sin(2wh)},\\
\widehat{\beta_2} &= \frac{w^2h^2\sin(3wh) - \sin(wh)h^2w^2 - 19h^2w^2\sin(2wh) - 12\sin(3wh) + 12\sin(wh) + 12\sin(2wh)}{4hw^2\sin(3wh) + 20hw^2\sin(wh) - 16hw^2\sin(2wh)},\\
\widehat{\beta_3} &= \frac{w^2h^2\sin(3wh) + 85\sin(wh)h^2w^2 + 28h^2w^2\sin(2wh) + 48wh\cos(wh) - 48wh + 24\sin(3wh) - 120\sin(wh) + 24\sin(2wh)}{24hw^2\sin(3wh) + 120hw^2\sin(wh) - 96hw^2\sin(2wh)},\\
\widehat{\beta_4} &= \frac{-34\sin(wh)h^2w^2 - h^2w^2\sin(2wh) - 24wh\cos(wh) + 24wh + 48\sin(wh) - 24\sin(2wh)}{24hw^2\sin(3wh) + 120hw^2\sin(wh) - 96hw^2\sin(2wh)}.
\end{aligned}
\tag{16}
$$

The converted coefficients of the method in series form is given as

$$
\begin{aligned}
\widehat{\beta_0} &= \frac{97}{1440}h + \frac{1}{640}hu^2 - \frac{439}{3628800}hu^4 - \frac{7211}{319334400}hu^6 + O(u^8),\\
\widehat{\beta_1} &= \frac{361}{360}h - \frac{29}{5040}hu^2 + \frac{17}{90720}hu^4 + \frac{1817}{39916800}hu^6 + O(u^8),\\
\widehat{\beta_2} &= \frac{37}{80}h + \frac{53}{6720}hu^2 + \frac{11}{67200}hu^4 - \frac{19}{17740800}hu^6 + O(u^8),\\
\widehat{\beta_3} &= -\frac{13}{360}h - \frac{1}{210}hu^2 - \frac{23}{56700}hu^4 - \frac{1}{22680}hu^6 + O(u^8),\\
\widehat{\beta_4} &= \frac{1}{288}h + \frac{43}{40320}hu^2 + \frac{91}{518400}hu^4 + \frac{7097}{}h^2u^6 + O(u^8).
\end{aligned}
\tag{17}
$$

Differentiating and evaluating at $x = x_n$ gives y'_{n+3}.

$$
\begin{aligned}
\widehat{\alpha_0} &= -\frac{1}{h},\\
\widehat{\alpha_1} &= \frac{1}{h},\\
\widehat{\beta_0} &= \frac{9h^2w^2\sin(2wh) + 2\sin(wh)h^2w^2 + 8wh\cos(wh) - 8wh - 16\sin(2wh) + 8\sin(3wh) + 8\sin(wh)}{32hw^2\sin(2wh) - 8hw^2\sin(3wh) - 40hw^2\sin(wh)},\\
\widehat{\beta_1} &= \frac{16h^2w^2\sin(2wh) - 27w^2h^2\sin(3wh) - 191\sin(wh)h^2w^2 - 24wh\cos(2wh) - 48wh\cos(wh)}{96hw^2 sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{-24\sin(wh) + 72wh + 120\sin(2wh) - 72\sin(3wh)}{96hw^2 sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_2} &= \frac{101h^2w^2 sin(2wh) - 11w^2h^2\sin(3wh) + 11\sin(wh)h^2w^2}{48hw^2\sin(2wh) - 12hw^2\sin(3wh) - 60hw^2\sin(wh)}\\
&\quad + \frac{36wh\cos(2wh) - 36wh - 36\sin(2wh) + 36\sin(3wh) - 36\sin(wh)}{48hw^2\sin(2wh) - 12hw^2\sin(3wh) - 60hw^2\sin(wh)},\\
\widehat{\beta_3} &= \frac{-16h^2w^2\sin(2wh) - 11w^2h^2\sin(3wh) - 175\sin(wh)h^2w^2 - 72wh\cos(2wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{48wh\cos(wh) + 24wh - 24\sin(2wh) - 24\sin(3wh) + 120\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_4} &= \frac{11h^2w^2\sin(2wh) + 38\sin(wh)h^2w^2 + 24wh\cos(2wh) - 24wh\cos(wh) + 24\sin(2wh) - 48\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}.
\end{aligned}
\tag{18}
$$

The converted coefficients of the method in series form is given as

$$\begin{aligned}
\widehat{\beta_0} &= \frac{119}{1440}h + \frac{571}{120960}hu^2 + \frac{1103}{3628800}hu^4 + \frac{8291}{319334400}hu^6 + O(u^8),\\
\widehat{\beta_1} &= \frac{9}{10}h - \frac{65}{6048}hu^2 - \frac{517}{907200}hu^4 - \frac{59}{1267200}hu^6 + O(u^8),\\
\widehat{\beta_2} &= \frac{263}{240}h + \frac{79}{20160}hu^2 - \frac{23}{201600}hu^4 - \frac{857}{53222400}hu^6 + O(u^8),\\
\widehat{\beta_3} &= \frac{4}{9}h + \frac{167}{30240}hu^2 + \frac{131}{181440}hu^4 + \frac{5431}{79833600}hu^6 + O(u^8),\\
\widehat{\beta_4} &= -\frac{11}{480}h - \frac{59}{17280}hu^2 - \frac{1241}{3628800}hu^4 - \frac{667}{21288960}h^2u^6 + O(u^8).
\end{aligned} \tag{19}$$

Differentiating and evaluating at $x = x_n$ gives y'_{n+4}.

$$\begin{aligned}
\widehat{\alpha_0} &= -\frac{1}{h},\\
\widehat{\alpha_1} &= \frac{1}{h},\\
\widehat{\beta_0} &= \frac{37h^2w^2\sin(2wh) + 10\sin(wh)h^2w^2 + 24wh\cos(2wh) - 24wh\cos(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{-48\sin(2wh) + 24\sin(3wh) + 24\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_1} &= \frac{-20h^2w^2\sin(2wh) - 37w^2h^2 sin(3wh) - 185\sin(wh)h^2w^2 - 48wh\cos(2wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{72wh\cos(wh) - 24wh\cos(3wh) + 120\sin(2wh) - 72\sin(3wh) - 24\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_2} &= \frac{121h^2w^2\sin(2wh) + 5w^2h^2\sin(3wh) - 5\sin(wh)h^2w^2 - 36wh\cos(wh)}{48hw^2\sin(2wh) - 12hw^2\sin(3wh) - 60hw^2\sin(wh)}\\
&\quad + \frac{36wh\cos(3wh) - 36\sin(2wh) + 36\sin(3wh) - 36\sin(wh)}{48hw^2\sin(2wh) - 12hw^2\sin(3wh) - 60hw^2\sin(wh)},\\
\widehat{\beta_3} &= \frac{20h^2w^2\sin(2wh) - 57w^2h^2\sin(3wh) - 205\sin(wh)h^2w^2 + 48wh\cos(2wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)}\\
&\quad + \frac{24wh\cos(wh) - 72wh\cos(3wh) - 24\sin(2wh) - 24\sin(3wh) + 120\sin(wh)}{96hw^2\sin(2wh) - 24hw^2\sin(3wh) - 120hw^2\sin(wh)},\\
\widehat{\beta_4} &= \frac{19h^2w^2\sin(2wh) - 10\sin(wh)h^2w^2 + 8wh\cos(3wh) - 8wh\cos(2wh) + 8\sin(2wh) - 16\sin(wh)}{32hw^2\sin(2wh) - 8hw^2\sin(3wh) - 40hw^2\sin(wh)}.
\end{aligned} \tag{20}$$

The converted coefficients of the method in series form is given as

$$\begin{aligned}
\widehat{\beta_0} &= \frac{9}{160}h + \frac{29}{120960}hu^2 - \frac{31}{145152}hu^4 - \frac{89}{3225600}hu^6 + O(u^8),\\
\widehat{\beta_1} &= \frac{377}{360}h - \frac{5}{432}hu^2 + \frac{73}{453600}hu^4 + \frac{2089}{39916800}hu^6 + O(u^8),\\
\widehat{\beta_2} &= \frac{35}{48}h + \frac{671}{20160}hu^2 + \frac{23}{28800}hu^4 + \frac{13}{1520640}hu^6 + O(u^8),\\
\widehat{\beta_3} &= \frac{161}{120}h - \frac{31}{945}hu^2 - \frac{139}{113400}hu^4 - \frac{53}{831600}hu^6 + O(u^8),\\
\widehat{\beta_4} &= \frac{469}{1440}h + \frac{1313}{120960}hu^2 + \frac{1741}{3628800}hu^4 + \frac{9721}{319334400}h^2u^6 + O(u^8).
\end{aligned} \tag{21}$$

3. Basic properties of the method

The implicit schemes from equation (6)-(13) derived are discrete schemes belonging to the class of LMMs of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{22}$$

Following Refs. [2, 11], we define the Local Truncation Error(LTE) associated with equation (22) by difference operator;

$$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x+jh) - h^2\beta_j y''(x+jh)]. \tag{23}$$

Expanding (23) by Taylor series, we have

$$L[y(x);h] = C_0 y(x) + C_1 hy'(x) + \cdots + C_q h^q y^{(q)}(x) + \cdots, \tag{24}$$

where

$$\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k,\\
C_1 &= \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k,\\
C_2 &= \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \cdots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k),\\
&\vdots\\
C_p &= \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \cdots + k^p\alpha_k) - \frac{1}{(q-2)!}\\
&\quad (\beta_1 + 2^{q-2}\beta_2 + \cdots + k^{q-2}\beta_k), q \geq 3.
\end{aligned}$$

3.1. Order and error constant

The LMM (22) is said to be of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant. Equation (6) has order $p = 5$ and error constant given by $C_{p+2} = \frac{1}{240}$. Equation (8) has order $p = 5$ and error constant given by $C_{p+2} = \frac{1}{120}$, Equation (10) has order $p = 5$ and error constant given by $C_{p+2} = \frac{1}{120}$ and Equation (12) has order $p = 5$ and error constant given by $C_{p+2} = \frac{-191}{720}$.

3.2. Zero stability

Theorem 1. *Zero-stability [1, 11]: A block method is said to be zero stable if as $h \to 0$, the roots r_j, $j = 1(1)k$ of the first characteristic polynomials $\rho(r) = 0$ that is*

$$\rho(r) = det\left[\sum A^{(0)}R^{k-1}\right] = 0,$$

satisfying $|R| \leq 1$, must be simple.
$\rho(z) = det[zA^{(0)} - A^{(i)} = 0]$,

$$\left[z\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right] = 0$$

$$\left[\begin{pmatrix}z&0&0&0\\0&z&0&0\\0&0&z&0\\0&0&0&z\end{pmatrix} - \begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right] = 0$$

$$z^3(z-1) = 0, \;\; z = 1,0,0,0$$

since $| z = 1,0,0,0 | \leq 1$, the method is stable.

3.3. Consistency

According to Ref. [12], a numerical method is said to be consistent if the following conditions are satisfied.

(i) The order of the method must be greater or equal to 1. i.e. $p \geq 1$;

(ii) $\Sigma_{j=0}^{k}\alpha_j = 0$;

(iii) $\rho(r) = \rho'(r) = 0$;

(iv) $\rho''(r) = 2!\sigma(r)$.

where ρ and σ are the first and second characteristics polynomial of equation (5). When these conditions are applied to the main scheme, it is found to be consistent.

Table 1: Numerical results and the absolute error for test problem 1.

x	Exact	Computed	Error
0.1	-0.10517091807564762	-0.10517091807437191	1.27571E-12
0.2	-0.22140275816016983	-0.22140275815480269	5.36714E-12
0.3	-0.34985880757600310	-0.34985880756315065	1.285245E-11
0.4	-0.49182469764127032	-0.49182469761687007	2.440025E-11
0.5	-0.64872127070012815	-0.64872127065934589	4.078226E-11
0.6	-0.82211880039050897	-0.82211880032762140	6.288757E-11
0.7	-1.01375270747047652	-1.01375270737873767	9.173885E-11
0.8	-1.22554092849246760	-1.22554092836395704	1.2851056E-10
0.9	-1.45960311115694966	-1.45960311098240009	1.7454957E-10
1.0	-1.71828182845904524	-1.71828182822764679	2.3139845E-10

Table 2: The comparison of error in new method and some existing methods for test problem 1.

x	Error in new method	Error in Ref. [13]	Error in Ref. [14]	Error in Ref. [15]
0.1	1.27571E-12	-	6.899835E-11	4.462679E-11
0.2	5.36714E-12	3.46017000E-09	1.525099E-10	9.864032E-11
0.3	1.285245E-11	5.67600300E-09	2.528244E-10	1.635218E-10
0.4	2.440025E-11	7.64127000E-09	3.725524E-10	2.409591E-10
0.5	4.078226E-11	1.04971347E-08	5.146680E-10	3.328765E-10
0.6	6.288757E-11	1.44950355E-08	6.825557E-10	4.414623E-10
0.7	9.173885E-11	1.87822380E-08	8.800651E-10	5.692067E-10
0.8	1.2851056E-10	2.27988702-08	1.111569E-09	7.189380E-10
0.9	1.7454957E-10	2.82582100E-08	1.382034E-09	8.938681E-10
1.0	2.3139845E-10	3.55473540E-08	1.697095E-09	1.097642E-09

Table 3: Numerical results and the absolute error for test problem 2.

x	Exact	Computed	Error
0.01	1.01979867335991086	1.01979867335968196	2.2890E-13
0.02	1.03918944084761210	1.03918944084696135	6.5075E-13
0.03	1.05816454641464877	1.05816454641402239	6.2638E-13
0.04	1.07671640027179207	1.07671640027015420	1.63787E-12
0.05	1.09483758192485392	1.09483758192097119	3.88273E-12
0.06	1.11252084314278561	1.11252084313620862	6.57699E-12
0.07	1.12975911085687365	1.12975911084716780	9.70585E-12
0.08	1.14654548998987291	1.14654548997661875	1.325416E-12
0.09	1.16287326621394559	1.16287326619673912	1.720647E-11
0.1	1.17873590863630285	1.17873590861475587	2.154698E-11

3.4. *Convergence*

Theorem 2. *[11]: The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Thus our block method is convergent since it is zero stable and consistent.*

4. Implementation and numerical examples

This section examines the performance of the new method by applying it to some second-order ODEs . The test problems outcomes are displayed in tabular form.
Error= $\left|\text{exact solution} - \text{computed solution}\right|$.

Test Problem 1:

$$y'' = y', \;\; y(0) = 1, \; y'(0) = 0, \; h = 0.1.$$

Exact solution: $y(x) = 1 - \exp(x)$.

Table 4: The comparison of error in new method and some existing methods for test problem 2.

x	Error in new method	Error in Ref. [16]
0.01	2.2890E-13	3.409E-11
0.02	6.5075E-13	3.239E-11
0.03	6.2638E-13	3.465E-11
0.04	1.63787E-12	2.40E-13
0.05	3.88273E-12	1.780E-12
0.06	6.57699E-12	7.467E-11
0.07	9.70585E-12	3.904E-11
0.08	1.325416E-12	4.132E-11
0.09	1.720647E-11	1.197E-10
0.1	2.154698E-11	8.342E-11

Table 5: Numerical results and the absolute error for test problem 3.

x	Exact	Computed	Error
0.01	0.904837418035959573	0.904837419827875764	1.791916191E-9
0.02	0.818730753077981859	0.818730757391773436	4.313791577E-9
0.03	0.740818220681717866	0.740818227549595599	6.867877733E-9
0.04	0.670320046035639301	0.670320054867431370	8.831792069E-9
0.05	0.606530659712633424	0.606530682641211184	2.2928577760E-8
0.06	0.548811636094026433	0.548811675353608366	3.9259581933E-8
0.07	0.496585303791409515	0.496585361588883748	5.7797474233E-8
0.08	0.449328964117221591	0.449329042672387956	7.8555166365E-8
0.09	0.406569659740599112	0.406569761324812764	1.01584213652E-7
0.1	0.367879441171442322	0.367879568145210102	1.26973767780E-7

Table 6: The comparison of error in new method and some existing methods for test problem 3.

x	Error in new method	Error in Ref. [17]
0.01	1.791916191E-9	1.29E-8
0.02	4.313791577E-9	3.01E-8
0.03	6.867877733E-9	5.04E-8
0.04	8.831792069E-9	9.32E-10
0.05	2.2928577760E-8	1.40E-7
0.06	3.9259581933E-8	1.90E-7
0.07	5.7797474233E-8	2.58E-7
0.08	7.8555166365E-8	3.32E-7
0.09	1.01584213652E-7	-
0.1	1.26973767780E-7	-

Test Problem 2:

$$y'' + \lambda^2 y = 0, \quad y(0) = 1, \; y'(0) = 2, \quad \lambda = 2.$$

Exact solution: $y(x) = \cos(2x) + \sin(2x)$.

Test Problem 3:

$$y'' - 100y = 0, \quad y(0) = 1, \; y'(0) = -10, \; h = 0.01.$$

Exact solution: $y(x) = \exp(-10x)$.

Table 1 shows the results generated for test problem 1 using the proposed method. Table 2 shows the comparison of the results of our method with that of Refs. [13–15]. It is obvious that our method is better in accuracy and also efficient than Refs. [13, 13, 15].

Table 3 represents the outcomes produced for test problem 2 utilizing the proposed method. Meanwhile, Table 4 demonstrates a favourable comparison of the proposed method with the one in Ref. [16].

Table 5 presents the results obtained for test problem 3 using the proposed method. Table 6 shows the comparison of the proposed method with Ref. [17]. It is observed that our method is more efficient and accurate than that of Ref. [17].

The implementation of the new scheme on three specific examples demonstrates its favourable comparison with exact solutions in Table 2 and Table 6 while the result in Table 4 competes well with the existing results.

5. Conclusion

Utilizing collocation and interpolation techniques, a novel class of continuous second derivative block methods for solving ODEs is constructed in this work. This innovative approach combines polynomial and trigonometric functions, implemented through code written in Maple, to develop an approximation solution. The resulting block techniques exhibit continuous coefficients and possess key properties of consistency, zero stability, and convergence. These characteristics contribute to a robust methodology that ensures reliability in solving ODEs, presenting a promising avenue for further exploration and application in this field.

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