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Robust-M new two-parameter estimator for linear regression models: Simulations and applications

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Abstract

In the presence of multicollinearity and outliers, the ordinary least squares estimator remains inconsistent and unreliable. Several estimators have been proposed that can co-handle the problems of multicollinearity and outliers simultaneously. However, there is still a need to explore some other robust methods when the two anomalies appear in the linear regression model and recommend it to end users of statistics. Therefore, this study proposed Robust-M New Two Parameter (RNTP) and examined its performance over some already existing ones in the presence of multicollinearity and outliers in the x-direction. The theoretical expression under some conditions was established to showcase the new estimator's superiority. A simulation study was carried out alongside some factors to show that the RNTP is better than all other estimators considered in the study. The simulation study results revealed that RNTP outperformed other estimators in the study using the minimum MSE as the criterion. Likewise, real-life data was applied to affirm this claim.

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1. Introduction

An explanation of a multiple linear regression model's matrix form is as follows:

$$y = X\beta + U_i, \tag{1}$$

such that y is an *(nx1)* vector endogenous variable, X is a complete design matrix of *(nxp)* exogenous variable, β is *(nx1)* unknown parameter which has *(px1)* vector and U_i is an *(nx1)* random error with $E(U_i) = 0$ and variance $V(U_i) = \sigma^2 I_n$ whereby σ^2 andI_n are unknown parameter and identity matrix of order n respectively.

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Equation (1) is commonly referred to as the Ordinary Least Squares Estimator (OLSE), which is defined as:

$$\widehat{\beta} = \eta^{-1} X^T y, \tag{2}$$

where $\eta = X^T X$.

When all of its assumptions are met, Equation (2) stays the best among all other unbiased estimators; otherwise, it becomes inefficient. The existence of extreme observations in the data is one of the reasons why OLSE becomes unsuitable for regression analysis. Meanwhile, the M-estimator [1] is the most commonly used approach for dealing with this problem. Others include the MM-estimator [2, 3], Least Trimmed Squares (LTS) [4, 5], S-estimator [6, 7], Least Absolute Deviation (LAD) [8, 9], and Least Quartile of Square (LQS) estimator [5]. Similarly, strong exogenous variable correlation reduces OLSE performance, which includes imprecise parameter estimation, a broad range of confidence intervals, and the development of a small t-ratio [7]. In order to address this issue in the literature, writers have developed certain biased estimators such as Ridge Estimator [8], Principal Component Estimator [9–11], coupled Ordinary Ridge Regression and OLS, and the suggested Liu estimator are examples of early biased estimators. Other estimators that can avoid the multicollinearity problem in linear regression models are two-parameter (TP), new two-parameter (NTP), modified ridge type (MRT), Kibria-Lukman (KL), Dawoud-Kibria (DK), and more recently, a generalized Kibria-Lukman (GKL) estimator [12] and a new ridge-type estimator [7], among others.

Outliers and multicollinearity however, may be unavoidable in a linear regression model, if this is the case researchers have developed various estimators that can deal with these two problems, such as the Ridge-M Estimator and the Ridge-MM Estimator. Meanwhile, some authors have combined the M, MM, LTS, and S estimators to offer robust ridge regression [13]. The M, MM, LTS, and S estimators, along with various other reliable regression estimators, were merged with the Liu estimator. Other methods are robust two-parameter (RTP) [14], robust Dawoud-Kibria (RDK) [15], and modified Ridge-M estimation [16]. Hence, to further explore robust methods of dealing with the problem of outliers and multicollinearity in linear regression analysis, this study proposes a robust-M new two-parameter estimator, especially when the anomaly is in the x-direction.

2. Some already existing Robust One and Two-Parameter estimators

2.1. Ridge-M regression

The ridge regression estimator was introduced by Hoerl & Kennard [8] since OLSE is inefficient in the presence of multicollinearity. This was done by introducing a biasing parameter k into the design matrix of η. Also, it was noted that the Ridge Regression Estimator (RRE) is always affected by outliers in the y-direction, which led Lukman *et al.* [17] to propose robust ridge regression, defined as:

$$\widehat{\beta}_K^M = \left(K\eta^{-1} + I\right)^{-1}\widehat{\beta}_M, \tag{3}$$

where $\widehat{\beta}_M = \min_\beta \sum_{i=1}^n \theta\left(\frac{u_i}{k}\right)$,such that $\widehat{\beta}_M$is the M-estimator, $k \geq 0$ and $u_i = y_i - x_i^T\widehat{\beta}_M$.

2.2. Robust Liu estimator

The Liu estimator is another biased estimator to handle the problem of multicollinearity in a linear regression model. It was introduced by Liu [11], which can be expressed as:

$$\widehat{\beta}_L = (\eta + I)^{-1}(X^T y + d\widehat{\beta}), \tag{4}$$

where *0<d<1*and d are the biasing parameters. Meanwhile, the Liu estimator has been noted to be affected by the extreme values, especially in the y-direction; this led Ref. [18] to propose its robust version, which can be defined as follows:

$$\widehat{\beta}_M^d = \left(\eta + I_p\right)^{-1}(\eta + \mathrm{dI}_p)\widehat{\beta}_M. \tag{5}$$

2.3. Robust Kibria-Lukman estimator

As an alternative to the one-biasing parameter estimator aside Ridge and Liu estimator, Ref. [19] proposed K-L estimator, defined as:

$$\widehat{\beta}_{\mathrm{KL}} = \left(\eta + \mathrm{kI}_p\right)^{-1}(\eta - \mathrm{kI}_p)\widehat{\beta}_{\mathrm{OLS}}. \tag{6}$$

The robust version of the K-L estimator when there are outliers in the y-direction was just recently proposed [20], and it is defined as:

$$\widehat{\beta}_M^{\mathrm{KL}} = \left(\eta + \mathrm{kI}_p\right)^{-1}(\eta - \mathrm{kI}_p)\widehat{\beta}_M. \tag{7}$$

2.4. Robust Two-Parameter estimator

In a bid to curtail the effect of multicollinearity in linear regression analysis, Ref. [21] came up with Two-parameter estimator. They defined the estimator as;

$$\widehat{\beta}_{\mathrm{TP}} = \left(\eta + \mathrm{kI}_p\right)^{-1}(\eta + \mathrm{kdI}_p)\widehat{\beta}_{\mathrm{OLS}}. \tag{8}$$

However, due to the sensitivity of the two-parameter estimator to outliers in the y-direction, Ref. [14] proposed a robust version expressed as:

$$\widehat{\beta}^{M}_{\mathrm{TP}} = \left(\eta + \mathrm{kI}_p\right)^{-1}(\eta + \mathrm{kdI}_p)\widehat{\beta}_{M}. \tag{9}$$

2.5. Robust Dawoud-Kibria estimator

As an alternative to already existing estimators that can deal with the problem of multicollinearity, Ref. [15] proposed the Dawoud-Kibria estimator, which was noted to outperform others under some conditions and simulation studies. The estimator can be expressed as:

$$\widehat{\beta}_{\mathrm{DK}} = \widehat{\beta}(\mathrm{DK}) = \left(\eta + k(1+d)I_p\right)^{-1}(\eta - k(1+d)I_p)\widehat{\beta}_{\mathrm{OLS}}. \tag{10}$$

Since the presence of extreme observations in the response variable direction has been noted to influence the performance of the Dawoud-Kibria estimator, Hence, to combat this problem, Ref. [15] proceeded and proposed a robust version of DKE by introducing $\widehat{\beta}_M$ instead of $\widehat{\beta}_{\mathrm{OLS}}$ used in the DKE. They defined the estimator as:

$$\widehat{\beta}_M(\mathrm{DK}) = \left(\eta + k(1+d)I_p\right)^{-1}(\eta - k(1+d)I_p)\widehat{\beta}_M. \tag{11}$$

3. Theoretical methodology of the proposed robust estimator

Yang and Chang (2010) proposed a new biased-based estimator as an alternative method of circumventing the problem of multicollinearity in regression analysis by following the methods of Refs. [11] and [22]. It is called the new two-parameter estimator and expressed as:

$$\widehat{\beta}_{(k,d)} = \left(\eta + I_p\right)^{-1}(\eta + \mathrm{dI}_p)\widehat{\beta}_K. \tag{12}$$

Sequel to the existence of extreme observation, especially in the x- variable direction, which has been noted to affect the new two-parameter estimator, a robust version of the new two-parameter estimator is hereby introduced and defined as:

$$\widehat{\beta}^{M}_{\mathrm{NTP}} = \left(\eta + I_p\right)^{-1}(\eta + \mathrm{dI}_p)\widehat{\beta}^{M}_K, \tag{13}$$

where $\widehat{\beta}^{M}_{K} = \left(\eta + k_M I_p\right)^{-1} X^T y$ such that $k_M = \frac{p\widehat{\sigma}^2_m}{\sum_{i=1}^{p}\widehat{\alpha}^2_{\mathrm{im}}}$,

3.1. The canonical form of Robust-M New Two Parameter (RNTP) estimator

Recall the general linear regression model as given in equation (1), therefore the canonical form is as follows:

$$y = S\gamma + U, \tag{14}$$

where $S = \mathrm{XM}$and$\gamma = M^T\beta$, M is the orthogonal matrix such that $S^TS = M^T\eta M = \mu =\mathrm{diag}(\mu_1, \mu_2, \mu_3, \ldots \mu_p)$where $\mu_1, \mu_2, \mu_3, \ldots \mu_p > 0$ are the ordered eigen values of η.

Let γ_m be the M-estimator of the equation $\sum_{i=1}^{n}\psi(\varepsilon_i/c) = 0$ and $\sum_{i=1}^{n}\psi(\varepsilon_i/c)\, s_i = 0$ such that $\varepsilon_i = y_i - s_i^T\widehat{\gamma}_m$ where c is a scale parameter and ψ is a selected useful function [23, 24]. Hence, the estimator in equation (14) can be expressed as:

$$\widehat{\gamma} = \mu^{-1}S^Ty, \tag{15}$$

$$\widehat{\gamma}_m = \min_{\alpha}\sum_{i=1}^{n}\theta\left(\frac{y_i - s_i^T\alpha}{k}\right), \tag{16}$$

$$\widehat{\gamma}(k) = \left(\eta + \mathrm{kI}_p\right)^{-1}S^Ty, \tag{17}$$

$$\widehat{\gamma}_m(k) = \left(I_p + \mathrm{kS}^{-1}\right)^{-1}\widehat{\gamma}_m, \tag{18}$$

$$\widehat{\gamma}_m(d) = \left(\eta + I_p\right)^{-1}\left(\eta + \mathrm{dI}_p\right)\widehat{\gamma}_m, \tag{19}$$

$$\widehat{\gamma}_m(\mathrm{KL}) = \left(\eta + k\, I_p\right)^{-1}\left(\eta - \mathrm{kI}_p\right)\widehat{\gamma}_m, \tag{20}$$

$$\widehat{\gamma}_m(\mathrm{TP}) = \left(\eta + k\, I_p\right)^{-1}\left(\eta + \mathrm{kdI}_p\right)\widehat{\gamma}_m, \tag{21}$$

$$\widehat{\gamma}_m(\mathrm{DK}) = \left(\eta + k(1+d)\, I_p\right)^{-1}\left(\eta - k(1+d)I_p\right)\widehat{\gamma}_m. \tag{22}$$

Consequently, the robust New Two-parameter estimator of $\gamma$is hereby defined as:

$$\widehat{\gamma}_m(\mathrm{NTP}) = (\eta + I)^{-1}\,(\eta + \mathrm{dI}))\widehat{\gamma}_m(k). \tag{23}$$

3.2. Determination of the MSE for Robust M New Two-Parameter (RNTP) estimator

Generally, the MSE of an OLS estimator $\widehat{\gamma}$ is expressed as:

$$\begin{aligned} \text{MSE}(\widehat{\gamma}) &= E(\widehat{\gamma}-\gamma)^T(\widehat{\gamma}-\gamma), \\ &= \text{tr}\,(\text{Cov}(\widehat{\gamma})) + \text{bias}(\widehat{\gamma})^T\,\text{bias}\,(\widehat{\gamma}). \end{aligned} \tag{24}$$

Equation (24) can also be given as:

$$\text{MSE}(\widehat{\gamma}) = \sum_{i=1}^{p}\frac{\sigma^2}{\mu_i}.$$

The MSE of robust M-estimator can be defined as:

$$\text{MSE}(\widehat{\gamma}_m) = \sum_{i=1}^{p}\Omega_{\text{ii}}, \tag{25}$$

where Ω_{ii}indicates the diagonal elements for $\text{Cov}(\widehat{\gamma}_m)$ which is equivalent to Ω that is finite.

MSE for robust version of Ridge estimator proposed by Ref. [8] can be expressed as:

$$\text{MSE}(\widehat{\gamma}_m(K)) = \sum_{i=1}^{p}\frac{\mu_i^2\Omega_{\text{ii}}}{(\mu_i+k)^2} + \sum_{i=1}^{p}\frac{k^2\widehat{\gamma}_i^2}{(\mu_i+k)^2}. \tag{26}$$

Liu M-estimator has the MSE of:

$$\text{MSE}(\widehat{\gamma}_m(d)) = \sum_{i=1}^{p}\frac{(\mu_i+d)^2\Omega_{\text{ii}}}{(\mu_i+1)^2} + \sum_{i=1}^{p}\frac{(1-d)\widehat{\gamma}_i^2}{(\mu_i+1)^2}. \tag{27}$$

Robust Kibria-Lukman has the following MSE:

$$\text{MSE}(\widehat{\gamma}_m(\text{KL})) = \sum_{i=1}^{p}\frac{(\mu_i+k)^2\Omega_{\text{ii}} + 4k^2\widehat{\gamma}_i^2\mu_i}{\mu_i(\mu_i+k)^2}. \tag{28}$$

Equation (29) is the MSE of robust TPE

$$\text{MSE}(\widehat{\gamma}_m(\text{TP})) = \sum_{i=1}^{p}\frac{(\mu_i+\text{kd})^2\Omega_{\text{ii}}}{(\mu_i+k)^2} + \sum_{i=1}^{p}\frac{k^2(1-d)^2\widehat{\gamma}_i^2}{(\mu_i+k)^2}. \tag{29}$$

MSE of robust Dawoud-Kibria is expressed as:

$$\text{MSE}(\widehat{\gamma}_m(\text{DK})) = \sum_{i=1}^{p}\frac{(\mu_i-k\,(1+d))^2\Omega_{\text{ii}}}{(\mu_i+k\,(1+d))^2} + \sum_{i=1}^{p}\frac{4k^2(1+d)^2\widehat{\gamma}_i^2}{(\mu_i+k\,(1+d))^2}. \tag{30}$$

Therefore, the MSE of the proposed robust New Two-Parameter Estimator (RNTPE) is as follows:

$$\text{MSE}(\widehat{\gamma}_m(\text{NTP})) = \sum_{i=1}^{p}\frac{(\mu_i+d)^2\mu_i\Omega_{\text{ii}}}{(\mu_i+1)^2(\mu_i+k)^2} + \sum_{i=1}^{p}\frac{((k+1-d)\,\mu_i+k)^2\widehat{\gamma}_i^2}{(\mu_i+1)^2(\mu_i+k)^2}. \tag{31}$$

3.3. Performance of the proposed Robust-M New Two-Parameter Estimator over some existing ones

The performance of the proposed estimator is established based on the following imposed conditions and theorems.

Conditions:

(i) The function ψ is skew symmetric and non-decreasing.

(ii) The $E(\varepsilon_i) = 0$ and $V(\varepsilon_i) = 1$. This simply means that the expected value of the residual is zero and variance is finite.

(iii) The diagonal element ofCov $(\widehat{\gamma}_m)$ is finite.

Theorem I

If $\sum_{i=1}^{p}\Omega_{\text{ii}} < \sum_{i=1}^{p}\sigma^2$then $\text{MSE}(\widehat{\gamma}_m(\text{NTP})) < \text{MSE}(\widehat{\gamma}(\text{NTP}))$,

Proof: The difference between the MSE of NTP and RNTP estimators.

$$D_{\text{RNTP}}^{\text{NTP}} = \sum_{i=1}^{p}\left[\frac{(\mu_i+d)^2\mu_i\Omega_{\text{ii}} + ((k+1-d)\mu_i+k)^2\gamma_i^2}{(\mu_i+1)^2(\mu_i+k)^2} - \frac{(\mu_i+d)\mu_i\sigma^2 + ((k+1-d)\mu_i+k)^2\gamma_i^2}{(\mu_i+1)^2(\mu_i+k)^2}\right], \tag{32}$$

$$= \sum_{i=1}^{p} \frac{(\mu_i + d)^2 \mu_i \Omega_{ii} - (\mu_i + d)^2 \mu_i \sigma^2}{(\mu_i + 1)^2 (\mu_i + k)^2}, \tag{33}$$

$$= \sum_{i-1}^{p} \frac{(\mu_i + d)^2 \mu_i \left(\Omega_{ii} - \sigma^2\right)}{(\mu_i + 1)^2 (\mu_i + k)^2}, \tag{34}$$

If $\left(\Omega_{ii} - \sigma^2\right) < 0$, therefore MSE $(\widehat{\gamma}_m(\text{NTP})) <$ MSE $(\widehat{\gamma}(\text{NTP}))$.

Hence, RNTP estimator is better than NTP, if $\sum_{i=1}^{p} \Omega_{ii} < \sum_{i=1}^{p} \sigma^2$.

Theorem II

MSE $(\widehat{\gamma}_m(\text{NTP})) <$ MSE $(\widehat{\gamma}_m(\text{DK}))$ if;

$$\sum_{i=1}^{p} M_i < \sum_{i=1}^{p} W_i,$$

Proof: The difference between the MSE of RNTP and RDK estimators.

$$D_{\text{RDK}}^{\text{RNTP}} = \sum_{i=1}^{p} \frac{(\mu_i + d_1)^2 \mu_i \Omega_{ii} + ((k_1 + 1 - d_1)\mu_i + k_1)^2 \gamma_i^2}{(\mu_i + 1)^2 (\mu_i + k_1)^2} - \sum_{i=1}^{p} \frac{(\mu_i - k_2(1 + d_2))^2 \Omega_{ii} + 4k_2^2 (1 + d_2)^2 \gamma_i^2}{(\mu_i + k_2(1 + d_2))^2}, \tag{35}$$

where $a = (\mu_i + 1)^2 (\mu_i + k_1)^2$, $a_1 = (\mu_i + d_1)^2 \mu_i$, $a_2 = ((k_1 + 1 - d_1)\mu_i + k_1)\gamma_i^2$, $b = (\mu_i + k_2(1 + d_2))^2$, $b_1 = (\mu_i - k_2(1 + d_2))^2$, $b_2 = 4k_2^2 (1 + d_2)^2 \gamma_i^2$. Such that $k_{1,}, k_2, d_1$, and$d_2$are the biasing parameters for RNTP and RDK respectively.

$$D_{\text{RDK}}^{\text{RNTP}} = \sum_{i=1}^{p} \left[\frac{a_1 \Omega_{ii} + a_2}{a} - \frac{b_1 \Omega_{ii} + b_2}{b} \right] \tag{36}$$

$$= \sum_{i=1}^{p} \left[\frac{a_1 b \Omega_{ii} + a_2 b - \text{ab}_1 \Omega_{ii} - \text{ab}_2}{\text{ab}} \right], \tag{37}$$

$$= \sum_{i=1}^{p} \left[\frac{(a_1 b - a_2 b) \Omega_{ii} - (\text{ab}_2 - a_2 b)}{\text{ab}} \right], \tag{38}$$

where $M_i = (a_1 b - a_2 b) \Omega_{ii}$, $W_i = (\text{ab}_2 - a_2 b)$. Therefore, the difference is less than zero if:

$$\sum_{i=1}^{p} M_i < \sum_{i=1}^{p} W_i.$$

3.3.1. *Selection of robust shrinkage parameters for the RNTP estimator*

Assume that $\widehat{\gamma}_m$ *N(0,1)*. This implies that $\widehat{\gamma}_m$ follows a Gaussian distribution with the same mean and covariance matrix equals $A^2 \eta^{-1}$. When $\sqrt{n}\left(\widehat{\gamma}_m^2 - \gamma\right)$ $N(0, A^2 \eta^{-1})$, then the assumption holds especially in practice, where $A^2 = \frac{c_0^2 E\left[\phi^2(\varepsilon/c_0)\right]}{E\left[\phi'(\varepsilon/c_0)\right]^2}$ such that c_0 is the scale estimate.

Also, according to Ref. [24], the unbiased estimator $\gamma_{\text{im}} = \widehat{\gamma}_{\text{im}}^2$ that is $E(\gamma_{\text{im}}) = \widehat{\gamma}_{\text{im}}$ and the unbiased estimator $\Omega_{ii} = A^2/\eta_i$ such that $A^2 = \frac{c^2 (n-p)^{-1} \sum_{i=1}^{n} \left[\phi(\varepsilon_i/c)\right]^2}{\sum_{i=1}^{n} \left[\frac{1}{n} \phi'(\varepsilon_i/c)\right]^2}$.

Biasing parameters for some robust estimators are hereby expressed as follows:

(i) Biasing parameter for Robust Ridge Regression (RRR) estimator by Ref. [17] is defined as:

$$\hat{K}_M = \frac{\text{pA}^2}{\sum_{i=1}^{p} \gamma_{\text{im}}^2}. \tag{39}$$

(ii) According to Ref. [18], they expressed parameter for robust Liu estimator as:

$$\hat{d}_m = 1 - A^2 \left[\frac{\sum_{i=1}^{p} \frac{1}{\eta_i(\eta_i + 1)}}{\sum_{i=1}^{p} \frac{\gamma_{\text{im}}^2}{(\eta_i + 1)^2}} \right]. \tag{40}$$

(iii) Biasing parameter for Robust Ridge-Liu estimator can be expressed as:

$$\hat{k}_m = \frac{1}{p}\sum_{i=1}^{p}\frac{A^2}{\widehat{\gamma}_{\text{im}}^2 - d\left(\frac{A^2}{\eta_i+\widehat{\gamma}_{\text{im}}^2}\right)} \tag{41}$$

$$\hat{d}_m = \min\left\{\frac{\widehat{\gamma}_{\text{im}}^2}{\frac{A^2}{\eta_i}+\widehat{\gamma}_{\text{im}}^2}\right\} \tag{42}$$

(iv) In respect to Ref. [15], they derived the biasing parameters for Robust Dawoud-Kibria as follows:

$$\hat{k}_m(\text{DK}) = \frac{1}{p}\sum_{i=1}^{p}\frac{A}{(1+d_{\text{TP}})\left(\frac{A}{\eta_i}+2\widehat{\gamma}_{\text{im}}^2\right)}\ , \tag{43}$$

where $d_{\text{TP}} = \min\left(\frac{\widehat{\gamma}_{\text{im}}^2}{\left(\frac{A}{\eta_i}\right)+\widehat{\gamma}_{\text{im}}^2}\right)_{i=1}^{p}$

Consequently, following Ref. [25], the biasing parameter for RNTP is as follows:

$$\hat{k}^{\text{RNTP}} = \frac{A^2(\eta_i+d)-(1-d)\,\eta_i\widehat{\gamma}_{\text{im}}}{(\eta_i+1)\,\widehat{\gamma}_{\text{im}}}, \tag{44}$$

such that $d = 0 < d < 1$ and

$$\hat{d}^{\text{RNTP}} = \frac{\sum_{i=1}^{p}\frac{\left(\widehat{\gamma}_{\text{im}}^2-A^2\right)}{(\eta_i+1)^2}}{\sum_{i=1}^{p}\frac{\left(A^2+\eta_i\widehat{\gamma}_{\text{im}}^2\right)}{(\eta_i+1)^2\eta_i}}. \tag{45}$$

3.4. Monte Carlo Experiment

To show the performance of the proposed RNTP over already existing estimators OLS, KL, DK, M, Ridge-M, Liu-M, KL-M, and Dk-M, a Monte Carlo experiment was conducted with the aid of R-statistical programming codes.

3.4.1. Procedures

The Monte Carlo experiment of the study was conducted using R statistical programming codes. Some of the estimators whose performances were compared are OLS, KL, DK, M, Ridge-M, Liu-M, KL-M, DK-M, and RNTP. All the exogenous variables were generated using the equation given in (46) and used by Ref. [26] among all other researchers.

$$x_{\text{ij}} = (1-\rho^2)^{\frac{1}{2}} z_{\text{ij}} + \rho z_{\text{ip}+1}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, p, \tag{46}$$

where z_{ij} are independent standard normal pseudo-random numbers and ρmeans the correlation between any two exogenous variables. Five (5) levels of different correlations were considered which are 0, 0.8, 0.9, 0.95, and 0.99 so as to exhibit the degrees of correlations between the explanatory variables. Meanwhile, the number of exogenous variables is $p = 3$ and expressed in a standardized form. Also, 10% and 20% of x_2 was random selected and replaced with outliers using $X_{(i)outlier}$ = Mo*Max (X_i) + X_i The magnitude of outliers (Mo) considered are (0, 5, and 10). Likewise, the response variable was generated using the following equations:

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_{\text{ip}} + e_i\ , \ i = 1, \ldots, p, \tag{47}$$

where $e_i \sim \text{iidN}(0, \sigma^2)$, whereby zero intercept was assumed for the model in (47), and the values of β was chosen to satisfy the constraints $\beta^T\beta = 1$suggested by Ref. [13]. The simulation study was replicated 1000 times for the sample sizes n= 20, 50, 100, and 250, respectively, with error variances (1, 5, and 10). Furthermore, the estimated MSE for each of the estimators was obtained for each replicate as in equation (48).

$$\text{MSE}(\beta) = \frac{1}{1000}\sum_{i=1}^{1000}(\beta - \beta)^2. \tag{48}$$

In the same vein, individual Ridge parameters of the estimators considered in the study were used.

4. Presentation of sample of Simulation results and discussion

4.1. Sample of simulation results

Sample of simulation results are presented in Tables 1 – 4 and graphically displayed in Figures 1 – 6.

Table 1: MSEs of estimators when $M_o = 0$ (No outlier).

			OLS	KL	DK	M	Ridge-M	Liu-M	KL-M	DK-M	NTP-M
n=20	σ=1	rho=0	0.3189	0.2891	0.2921	0.3333	0.2843	**0.2778**	0.2887	0.2913	0.3391
		rho=0.8	0.9855	0.3836	0.6545	1.0354	0.4233	0.5406	0.383	0.6519	**0.2965**
		rho=0.9	1.8789	0.5703	1.1132	1.9754	0.659	0.7135	0.5695	1.1104	**0.4679**
		rho=0.95	3.6694	0.9365	1.9543	3.8588	1.1255	0.9946	0.9394	1.9494	**0.8677**
		rho=0.99	18	3.8902	8.3505	18.934	4.8246	**3.0628**	3.92	8.3318	4.3981
	σ=5	rho=0	7.9715	3.1457	4.6774	8.3336	3.1959	6.6965	3.1488	4.6762	**1.6928**
		rho=0.8	24.637	6.6164	12.286	25.886	7.574	13.129	6.6571	12.276	**4.2038**
		rho=0.9	46.974	11.37	22.043	49.384	13.466	16.809	11.446	21.983	**8.6176**
		rho=0.95	91.734	20.703	41.843	96.47	25.133	23.032	20.87	41.78	**18.491**
		rho=0.99	450	94.831	201.16	473.35	117.72	**74.783**	95.699	200.71	107
	σ=10	rho=0	31.886	11.269	17.781	33.334	11.536	26.758	11.295	17.779	**4.9457**
		rho=0.8	98.546	26.056	48.319	103.54	29.722	52.511	26.228	48.232	**16.042**
		rho=0.9	187.89	45.125	87.19	197.54	53.39	67.138	45.453	86.864	**33.904**
		rho=0.95	366.94	82.508	166.34	385.88	100.11	91.928	83.188	166.05	**73.465**
		rho=0.99	1800	379.04	803.77	1893.4	470.54	**299.28**	382.53	801.53	427.64
n=50	σ=1	rho=0	0.0983	0.0861	0.0931	0.1031	0.0864	0.0933	0.0861	0.093	**0.0846**
		rho=0.8	0.2793	0.1607	0.2219	0.2907	0.1673	0.2241	0.1604	0.2214	**0.1118**
		rho=0.9	0.5264	0.2251	0.3772	0.5473	0.2452	0.3515	0.2243	0.3762	**0.1495**
		rho=0.95	1.0216	0.3218	0.6444	1.0617	0.3726	0.5134	0.3208	0.6414	**0.2295**
		rho=0.99	4.986	1.0442	2.3695	5.1796	1.3089	0.8027	1.0437	2.3575	**1.006**
	σ=5	rho=0	2.4579	1.0747	1.5736	2.5763	1.1208	2.3292	1.0777	1.5705	**0.6855**
		rho=0.8	6.9836	1.9255	3.5974	7.2673	2.1955	5.5896	1.9306	3.5838	**0.9857**
		rho=0.9	13.159	3.1124	6.3099	13.682	3.6848	8.5679	3.1193	6.2949	**1.6652**
		rho=0.95	25.54	5.4184	11.549	26.542	6.6113	12.241	5.4306	11.483	**3.3848**
		rho=0.99	124.65	23.719	51.99	129.49	29.877	18.419	23.789	51.701	**22.35**
	σ=10	rho=0	9.8315	3.5137	5.6652	10.305	3.6307	9.3171	3.5426	5.6579	**1.4411**
		rho=0.8	27.934	7.27	13.488	29.069	8.2078	22.357	7.2999	13.478	**3.169**
		rho=0.9	52.637	12.086	24.25	54.726	14.237	34.264	12.124	24.195	**6.086**
		rho=0.95	102.16	21.375	45.222	106.17	26.021	48.85	21.436	45.002	**13.087**
		rho=0.99	498.6	94.611	207.32	517.96	119.14	**73.759**	94.907	206.02	89.066
n=100	σ=1	rho=0	0.0427	**0.0374**	0.0395	1.125	0.6693	1.0494	0.676	0.8007	0.6093
		rho=0.8	0.1765	0.1205	0.1542	0.1382	0.0935	0.1177	0.0918	0.1131	**0.0622**
		rho=0.9	0.3468	0.1877	0.2834	0.2624	0.1416	0.1982	0.1346	0.1954	**0.0824**
		rho=0.95	0.6876	0.2565	0.4288	0.5111	0.2115	0.3134	0.1899	0.3396	**0.1175**
		rho=0.99	3.4145	0.6295	1.8025	2.5017	0.6436	0.6947	0.5071	1.2449	**0.4343**
	σ=5	rho=0	1.0665	**0.5621**	0.6676	1.125	0.6693	1.0494	0.676	0.8007	0.6093
		rho=0.8	4.4116	0.8793	2.5973	3.4554	1.0414	2.9404	0.8965	1.7658	**0.4477**
		rho=0.9	8.6696	1.5078	5.114	6.5589	1.7004	4.953	1.4041	2.95	**0.6599**
		rho=0.95	17.19	2.5012	6.7989	12.777	3.016	7.765	2.4142	5.3312	**1.2607**
		rho=0.99	85.362	11.092	31.401	62.544	13.452	16.328	10.447	24.763	**8.4291**

		rho=0	4.2661	1.4531	1.9	4.4999	1.7867	4.1984	1.7889	2.6067	**0.9561**
		rho=0.8	17.647	2.9698	8.9953	13.822	3.7091	11.762	3.2188	6.3105	**1.311**
	σ=10	rho=0.9	34.678	5.331	18.762	26.236	6.3925	19.812	5.3107	10.897	**2.2842**
		rho=0.95	68.76	9.4807	30.31	51.108	11.684	31.047	9.3918	20.317	**4.7198**
		rho=0.99	341.45	43.85	118.05	250.17	53.467	65.248	41.556	97.982	**33.374**
		rho=0	0.0151	0.0151	**0.015**	0.0158	0.0151	**0.015**	0.0151	**0.015**	0.0162
		rho=0.8	0.0415	0.0369	0.0391	0.0433	0.037	0.0402	0.0369	0.0391	**0.0306**
	σ=1	rho=0.9	0.0774	0.0616	0.0698	0.0806	0.0619	0.0724	0.0616	0.0698	**0.0446**
		rho=0.95	0.1494	0.0984	0.1258	0.1554	0.1005	0.1304	0.0984	0.1256	**0.0616**
		rho=0.99	0.7257	0.2391	0.4775	0.7543	0.2749	0.4004	0.2396	0.4754	**0.1494**
		rho=0	0.3765	0.3213	0.3372	0.3952	**0.3207**	0.3736	0.321	0.3371	0.4005
		rho=0.8	1.0378	0.3971	0.6889	1.0815	0.4378	1.0048	0.3988	0.6874	**0.2196**
n=250	σ=5	rho=0.9	1.9357	0.5577	1.132	2.0146	0.6525	1.8096	0.5607	1.1316	**0.2569**
		rho=0.95	3.7357	0.8579	1.9146	3.885	1.0604	3.2592	0.8648	1.9016	**0.3665**
		rho=0.99	18.144	3.3114	7.8764	18.858	4.329	9.815	3.3556	7.8411	**1.8573**
		rho=0	1.5061	0.856	1.0815	1.581	0.8584	1.4943	0.859	1.0856	**0.6889**
		rho=0.8	4.1513	1.2023	2.2618	4.326	1.3728	4.019	1.2128	2.259	**0.5588**
	σ=10	rho=0.9	7.743	1.8544	3.8171	8.0583	2.2195	7.2384	1.872	3.8387	**0.7597**
		rho=0.95	14.943	3.0969	6.8057	15.54	3.8672	13.037	3.1323	6.7854	**1.2425**
		rho=0.99	72.575	12.963	30.503	75.431	16.965	39.257	13.153	30.399	**7.2084**

Table 2: MSEs of estimators when $P_o = 10\%$ and $M_o = 5$.

			OLS	KL	DK	M	Ridge-M	Liu-M	KL-M	DK-M	NTP-M
	σ=1	rho=0	0.2429	0.1539	0.2018	0.25228	0.1571	0.2034	0.153	0.201	**0.1057**
		rho=0.8	0.5449	0.2065	0.3223	0.58062	0.2365	0.3258	0.2061	0.322	**0.1472**
		rho=0.9	1.056	0.2761	0.4726	1.0914	0.3455	0.4088	0.2743	0.469	**0.2219**
		rho=0.95	1.795	0.355	0.7221	1.91128	0.4807	0.5064	0.3595	0.72	**0.3325**
		rho=0.99	8.511	**1.2643**	2.9546	9.04265	1.743	1.7129	1.297	2.941	1.5048
	σ=5	rho=0	6.0733	1.9046	3.1681	6.30708	2.0691	5.0793	1.8953	3.15	**0.9045**
n=20		rho=0.8	13.622	3.3287	6.4767	14.5155	3.8335	8.1171	3.3731	6.483	**1.9715**
		rho=0.9	26.401	5.6584	10.578	27.285	6.6198	10.112	5.7395	10.48	**4.0117**
		rho=0.95	44.874	8.0291	16.165	47.7821	10.033	12.138	8.2838	16.18	**6.5727**
		rho=0.99	212.77	**31.674**	73.777	226.066	41.851	43.019	32.679	73.5	36.016
	σ=10	rho=0	24.293	7.1676	12.596	25.2283	7.745	20.316	7.1417	12.52	**3.0965**
		rho=0.8	54.487	13.075	25.875	58.0618	14.854	32.476	13.277	25.91	**7.3414**
		rho=0.9	105.6	22.551	42.178	109.14	26.09	40.419	22.902	41.76	**15.663**
		rho=0.95	179.5	32.078	64.472	191.128	39.772	48.449	33.127	64.6	**25.867**
		rho=0.99	851.1	**126.73**	295.11	904.265	167.14	172.22	130.79	294.1	143.79
	σ=1	rho=0	0.0816	0.0678	0.0773	0.08512	0.068	0.0773	0.0678	0.077	**0.0537**
		rho=0.8	0.2084	0.1113	0.1612	0.22223	0.1193	0.164	0.1116	0.162	**0.0732**
		rho=0.9	0.3586	0.1283	0.2114	0.36926	0.1508	0.2215	0.1273	0.212	**0.0807**
		rho=0.95	0.5532	0.1525	0.2658	0.59536	0.196	0.277	0.1537	0.267	**0.1061**
		rho=0.99	2.8696	**0.4296**	0.9915	3.08895	0.6326	0.6697	0.4427	1.001	0.4548
	σ=5	rho=0	2.0407	0.7075	1.1436	2.12809	0.7623	1.9325	0.7116	1.141	**0.2507**
		rho=0.8	5.2103	1.258	2.4245	5.55573	1.4931	4.0961	1.2852	2.445	**0.6057**

n=50		rho=0.9	8.9661	1.7417	3.5129	9.2314	2.1406	5.529	1.7515	3.474	**0.8882**
		rho=0.95	13.83	2.5084	5.0337	14.8841	3.1898	6.8985	2.5738	5.077	**1.55**
		rho=0.99	71.74	10.71	24.747	77.2237	14.225	16.717	11.143	25	**10.27**
	σ=10	rho=0	8.1629	2.3605	4.3373	8.51236	2.5939	7.73	2.381	4.333	**0.7816**
		rho=0.8	20.841	4.8083	9.6879	22.2229	5.5901	16.383	4.9273	9.774	**2.178**
		rho=0.9	35.865	6.8666	14.025	36.9256	8.2267	22.112	6.9236	13.87	**3.3449**
		rho=0.95	55.32	9.9958	20.127	59.5362	12.448	27.583	10.27	20.29	**5.9946**
		rho=0.99	286.96	42.911	98.97	308.895	56.666	66.874	44.66	100	**40.912**
		rho=0	0.0294	**0.0269**	0.0287	0.76999	0.3512	0.721	0.3376	0.495	0.1427
	σ=1	rho=0.8	0.1591	0.1079	0.1413	0.06293	0.0481	0.0565	0.0476	0.056	**0.0366**
		rho=0.9	0.2001	0.0951	0.1566	0.11545	0.071	0.095	0.0678	0.094	**0.0448**
		rho=0.95	0.7933	0.2433	0.4426	0.24056	0.1039	0.1589	0.0895	0.152	**0.0538**
		rho=0.99	3.783	0.4902	1.4086	1.05466	0.2469	0.3083	0.163	0.332	**0.1349**
		rho=0	0.7343	0.2302	0.4122	0.76999	0.3512	0.721	0.3376	0.495	**0.1427**
	σ=5	rho=0.8	3.9786	0.4949	1.3559	1.57329	0.5121	1.4117	0.4392	0.707	**0.2084**
		rho=0.9	5.0032	0.2295	1.6226	2.88617	0.7237	2.3758	0.5787	1.083	**0.247**
n=100		rho=0.95	19.833	1.6313	7.3716	6.01405	1.2929	3.9687	1.0187	2.072	**0.4918**
		rho=0.99	94.576	7.2558	33.416	26.3665	4.7526	7.7307	3.7053	7.973	**2.5968**
		rho=0	2.9372	0.494	1.2814	3.07996	1.0361	2.884	0.9612	1.589	**0.3692**
	σ=10	rho=0.8	15.915	1.5015	5.1724	6.29316	1.706	5.6467	1.4896	2.659	**0.6149**
		rho=0.9	20.013	0.741	6.2265	11.5447	2.5801	9.5032	2.1714	4.308	**0.8434**
		rho=0.95	79.331	6.0236	29.817	24.0562	4.902	15.873	4.0608	8.28	**1.8246**
		rho=0.99	378.3	28.517	132.76	105.466	18.796	30.938	14.912	31.89	**10.261**
		rho=0	0.0121	0.0117	0.012	0.01271	0.0117	0.012	0.0117	0.012	**0.0112**
	σ=1	rho=0.8	0.0247	0.0222	0.024	0.02608	0.0222	0.024	0.0222	0.024	**0.0193**
		rho=0.9	0.0435	0.0348	0.0407	0.04534	0.0351	0.0408	0.0347	0.041	**0.0269**
		rho=0.95	0.0772	0.0506	0.0675	0.0811	0.0523	0.0679	0.0506	0.068	**0.0339**
		rho=0.99	0.3601	0.0944	0.1839	0.37881	0.1246	0.1991	0.0951	0.185	**0.0567**
		rho=0	0.3024	0.1825	0.2482	0.31783	0.186	0.2999	0.1831	0.248	**0.076**
	σ=5	rho=0.8	0.6165	0.2276	0.3534	0.65196	0.2594	0.599	0.229	0.354	**0.0907**
		rho=0.9	1.0865	0.2733	0.4648	1.13349	0.3437	1.0203	0.272	0.463	**0.1133**
n=250		rho=0.95	1.9306	0.359	0.703	2.02755	0.4953	1.6967	0.3681	0.707	**0.1622**
		rho=0.99	9.0021	1.2274	2.8542	9.47018	1.745	4.9803	1.2925	2.87	**0.6957**
		rho=0	1.2096	0.4642	0.7185	1.2713	0.4953	1.1998	0.4689	0.725	**0.1513**
	σ=10	rho=0.8	2.4659	0.6447	1.1095	2.60784	0.7609	2.3958	0.6542	1.119	**0.2298**
		rho=0.9	4.3461	0.896	1.7354	4.53394	1.1024	4.0812	0.898	1.719	**0.3327**
		rho=0.95	7.7223	1.3295	2.8026	8.11021	1.7302	6.7866	1.3795	2.819	**0.5411**
		rho=0.99	36.008	4.9455	11.417	37.8807	6.7709	19.922	5.229	11.48	**2.6928**

Table 3: MSEs of estimators when $P_o = 10\%$ and $M_o = 10$.

			OLS	KL	DK	M	Ridge-M	Liu-M	KL-M	DK-M	NTP-M
	σ=1	rho=0	0.2389	0.1492	0.1983	0.2482	0.1526	0.1992	0.1483	0.1979	**0.0986**
		rho=0.8	0.5338	0.1954	0.3098	0.57	0.2257	0.3151	0.1951	0.3101	**0.1324**
		rho=0.9	1.0474	0.2665	0.4435	1.0827	0.3363	0.3959	0.2646	0.4414	**0.2098**
		rho=0.95	1.7789	0.3392	0.6333	1.8947	0.4654	0.4709	0.3436	0.6342	**0.3068**
		rho=0.99	8.4996	**1.2528**	2.9367	9.0331	1.7318	1.7071	1.2861	2.9268	1.4874

n=20	σ=5	rho=0	5.9717	1.7879	3.059	0.7592	0.3365	0.7112	0.3233	0.4839	**0.1053**
		rho=0.8	13.345	3.1047	6.2155	1.5417	0.4842	1.384	0.4117	0.6716	**0.1566**
		rho=0.9	26.184	5.4522	10.269	2.8473	0.701	2.3411	0.5556	1.0473	**0.2254**
		rho=0.95	44.472	7.6971	15.668	5.9811	1.2721	3.9449	0.9995	2.0506	**0.4659**
		rho=0.99	212.49	31.395	73.417	26.333	4.7308	7.7062	3.6869	7.9429	**2.5741**
	σ=10	rho=0	23.887	6.7264	12.218	24.824	7.3515	19.91	6.7055	12.141	**2.7947**
		rho=0.8	53.378	12.212	24.862	57.003	14.048	31.454	12.427	24.905	**6.791**
		rho=0.9	104.74	21.738	41.044	108.27	25.331	39.473	22.108	40.652	**15.11**
		rho=0.95	177.89	30.763	62.695	189.47	38.523	46.919	31.794	62.737	**24.873**
		rho=0.99	849.96	**125.62**	293.67	903.31	166.1	170.92	129.7	292.74	142.83
n=50	σ=1	rho=0	0.0814	0.0674	0.077	0.085	0.0675	0.077	0.0674	0.077	**0.0529**
		rho=0.8	0.2074	0.1088	0.1596	0.2214	0.117	0.1619	0.1091	0.1598	**0.0708**
		rho=0.9	0.3556	0.1264	0.21	0.3661	0.1488	0.219	0.1253	0.2103	**0.0788**
		rho=0.95	0.5561	0.1502	0.2629	0.5986	0.1944	0.2747	0.1514	0.2638	**0.1037**
		rho=0.99	2.8817	**0.4293**	0.9859	3.102	0.6329	0.6625	0.4427	0.9961	0.453
	σ=5	rho=0	2.035	0.6896	1.1239	2.1241	0.7453	1.9255	0.694	1.1222	**0.2168**
		rho=0.8	5.1861	1.2109	2.3596	5.5345	1.4488	4.0451	1.2398	2.3815	**0.5479**
		rho=0.9	8.8909	1.6979	3.4405	9.1535	2.093	5.473	1.7052	3.4048	**0.8351**
		rho=0.95	13.903	2.4756	4.993	14.966	3.1604	6.8635	2.5424	5.037	**1.5062**
		rho=0.99	72.042	10.711	24.638	77.549	14.24	16.565	11.154	24.896	**10.267**
	σ=10	rho=0	8.14	2.2903	4.276	8.4963	2.53	7.702	2.3125	4.281	**0.6958**
		rho=0.8	20.744	4.635	9.4384	22.138	5.4274	16.18	4.7617	9.5267	**2.0322**
		rho=0.9	35.563	6.6965	13.757	36.614	8.0452	21.891	6.7443	13.615	**3.1938**
		rho=0.95	55.613	9.8725	19.976	59.865	12.341	27.452	10.152	20.153	**5.8785**
		rho=0.99	288.17	42.912	98.547	310.2	56.731	66.261	44.703	99.581	**40.926**
n=100	σ=1	rho=0	0.0284	0.0259	0.0277	0.7592	0.3365	0.7112	0.3233	0.4839	**0.1053**
		rho=0.8	0.1576	0.1078	0.1408	0.0617	0.047	0.0554	0.0465	0.0553	**0.0355**
		rho=0.9	0.2025	0.0962	0.1599	0.1139	0.0697	0.0936	0.0666	0.0929	**0.0439**
		rho=0.95	0.7848	0.2449	0.4497	0.2392	0.103	0.1578	0.0887	0.152	**0.0531**
		rho=0.99	3.7659	0.492	1.2988	1.0533	0.2461	0.3082	0.1622	0.3295	**0.1331**
	σ=5	rho=0	0.7089	0.2161	0.4028	0.7592	0.3365	0.7112	0.3233	0.4839	**0.1053**
		rho=0.8	3.9407	0.5016	1.2185	1.5417	0.4842	1.384	0.4117	0.6716	**0.1566**
		rho=0.9	5.0614	0.1921	1.4883	2.8473	0.701	2.3411	0.5556	1.0473	**0.2254**
		rho=0.95	19.621	1.6077	7.5364	5.9811	1.2721	3.9449	0.9995	2.0506	**0.4659**
		rho=0.99	94.147	7.1721	32.407	26.333	4.7308	7.7062	3.6869	7.9429	**2.5741**
	σ=10	rho=0	2.8355	0.442	1.1981	3.0368	0.9756	2.845	0.9013	1.5239	**0.2623**
		rho=0.8	15.763	1.4586	4.8987	6.1667	1.6025	5.536	1.3905	2.5532	**0.4858**
		rho=0.9	20.246	0.5696	5.8606	11.389	2.4936	9.3643	2.0871	4.1844	**0.7735**
		rho=0.95	78.485	5.8683	30.146	23.924	4.8208	15.78	3.9889	8.2022	**1.7458**
		rho=0.99	376.59	28.112	129.6	105.33	18.709	30.826	14.839	31.772	**10.186**
	σ=1	rho=0	0.012	0.0116	0.0119	0.0126	0.0116	0.0119	0.0116	0.0119	**0.0111**
		rho=0.8	0.0245	0.022	0.0238	0.0259	0.0221	0.0238	0.022	0.0238	**0.0191**
		rho=0.9	0.0433	0.0346	0.0406	0.0451	0.0349	0.0406	0.0346	0.0406	**0.0267**
		rho=0.95	0.077	0.0504	0.0674	0.0809	0.0522	0.0677	0.0505	0.0674	**0.0338**
		rho=0.99	0.3599	0.0943	0.1845	0.3786	0.1244	0.199	0.0949	0.1853	**0.0565**
		rho=0	0.3004	0.1804	0.2466	0.3157	0.1839	0.2979	0.1809	0.2467	**0.0718**
		rho=0.8	0.6132	0.2247	0.3509	0.6483	0.2565	0.5957	0.2259	0.352	**0.0869**

	σ=5	rho=0.9	1.082	0.269	0.458	1.1285	0.3396	1.0158	0.2677	0.4558	**0.1088**
n=250		rho=0.95	1.9261	0.3549	0.6977	2.0226	0.4914	1.6922	0.3639	0.7022	**0.1578**
		rho=0.99	8.9977	1.2233	2.8496	9.4651	1.7408	4.9765	1.2881	2.8645	**0.6916**
		rho=0	1.2014	0.4544	0.7084	1.2627	0.4856	1.1916	0.459	0.7145	**0.1303**
		rho=0.8	2.4528	0.6333	1.0936	2.5931	0.7493	2.3828	0.6421	1.1029	**0.2126**
	σ=10	rho=0.9	4.3279	0.8802	1.7132	4.514	1.0869	4.0631	0.8823	1.697	**0.3142**
		rho=0.95	7.7046	1.3144	2.786	8.0905	1.715	6.7689	1.3641	2.8032	**0.5232**
		rho=0.99	35.991	4.9296	11.399	37.86	6.7542	19.906	5.2121	11.458	**2.6762**

Table 4: MSEs of estimators when P_o = 20% and M_o = 10.

			OLS	**KL**	**DK**	**M**	**Ridge-M**	**Liu-M**	**KL-M**	**DK-M**	**NTP-M**
	σ=1	rho=0	0.2187	0.135	0.1824	0.2277	0.13891	0.1822	0.1348	0.182	**0.0902**
		rho=0.8	0.3174	0.182	0.2487	0.3319	0.18793	0.2506	0.1812	0.2484	**0.1176**
		rho=0.9	0.4231	0.194	0.2876	0.4457	0.2103	0.2916	0.1937	0.2872	**0.1225**
		rho=0.95	1.2987	0.316	0.5416	1.3446	0.39807	0.4547	0.3097	0.5306	**0.2521**
		rho=0.99	5.1929	**0.854**	1.872	5.3992	1.15282	1.1264	0.8512	1.8313	0.9277
n=20	σ=5	rho=0	5.4678	1.538	2.9365	5.6919	1.70985	4.5431	1.5404	2.9171	**0.6647**
		rho=0.8	7.9356	2.282	4.1579	8.2972	2.5266	6.2522	2.2891	4.1496	**1.0344**
		rho=0.9	10.576	2.659	5.2063	11.142	3.02942	7.2739	2.6916	5.1701	**1.2871**
		rho=0.95	32.467	6.618	13.017	33.614	7.77986	11.344	6.558	12.723	**4.6954**
		rho=0.99	129.82	**21.12**	46.664	134.98	26.9222	28.083	21.262	45.686	21.654
	σ=10	rho=0	21.871	5.763	11.719	22.768	6.40137	18.171	5.7872	11.643	**2.4586**
		rho=0.8	31.742	8.68	16.631	33.189	9.62286	25.006	8.7257	16.58	**3.8389**
		rho=0.9	42.306	10.3	20.816	44.569	11.6873	29.092	10.447	20.667	**4.9512**
		rho=0.95	129.87	26.37	52.055	134.46	30.7535	45.38	26.157	50.894	**18.504**
		rho=0.99	519.29	**84.51**	186.62	539.92	107.404	112.3	85.096	182.74	86.378
		rho=0	0.0832	0.069	0.0786	0.0877	0.0687	0.0786	0.0685	0.0786	**0.0536**
	σ=1	rho=0.8	0.1778	0.102	0.1437	0.1872	0.10698	0.145	0.1014	0.1436	**0.0666**
		rho=0.9	0.3876	0.131	0.2203	0.4032	0.15618	0.2293	0.1295	0.2197	**0.0816**
		rho=0.95	0.4621	0.14	0.242	0.4885	0.17363	0.254	0.1397	0.2413	**0.0914**
		rho=0.99	2.3055	0.336	0.7718	2.4301	0.50262	0.535	0.3371	0.7678	**0.3339**
		rho=0	2.0801	0.7	1.1375	2.1918	0.7616	1.9645	0.7076	1.1427	**0.2227**
	σ=5	rho=0.8	4.4455	1.049	1.9878	4.679	1.24647	3.6256	1.0564	1.9901	**0.4303**
n=50		rho=0.9	9.691	1.829	3.6831	10.08	2.25611	5.73	1.8254	3.6438	**0.9136**
		rho=0.95	11.552	2.105	4.2236	12.212	2.64986	6.3445	2.1311	4.2033	**1.1746**
		rho=0.99	57.638	8.142	19.286	60.753	10.9356	13.384	8.3074	19.185	**7.3218**
		rho=0	8.3202	2.334	4.3291	8.7673	2.5959	7.8582	2.3706	4.3528	**0.7225**
	σ=10	rho=0.8	17.782	3.947	7.9498	18.716	4.60811	14.502	3.9898	7.959	**1.567**
		rho=0.9	38.764	7.23	14.731	40.32	8.70642	22.92	7.2339	14.574	**3.5124**
		rho=0.95	46.208	8.36	16.901	48.848	10.294	25.377	8.4829	16.819	**4.5573**
		rho=0.99	230.55	32.61	77.143	243.01	43.5064	53.544	33.3	76.737	**29.149**
		rho=0	0.033	0.03	0.0322	0.0323	0.02835	0.0301	0.0283	0.0301	**0.0253**
		rho=0.8	0.1708	0.115	0.1518	0.0635	0.04829	0.057	0.0477	0.0569	**0.0363**
	σ=1	rho=0.9	0.2737	0.127	0.2117	0.1169	0.06928	0.094	0.0659	0.0932	**0.0433**
		rho=0.95	0.6474	0.19	0.3776	0.2262	0.10059	0.1534	0.0873	0.1482	**0.0527**

		rho=0.99	3.3579	0.363	1.0676	1.0038	0.23371	0.2947	0.1521	0.3137	**0.1228**
		rho=0	0.8257	0.256	0.4538	0.8072	0.34912	0.752	0.334	0.5038	**0.1048**
		rho=0.8	4.271	0.562	1.3321	1.5882	0.4945	1.425	0.4227	0.6863	**0.1518**
	σ=5	rho=0.9	6.8413	0.354	2.2362	2.9236	0.69692	2.3496	0.5593	1.0378	**0.2225**
n=100		rho=0.95	16.184	0.822	5.3649	5.6545	1.21364	3.8355	0.958	1.9537	**0.4309**
		rho=0.99	83.948	4.546	26.489	25.096	4.42133	7.3688	3.4269	7.5284	**2.3064**
		rho=0	3.3029	0.563	1.5963	3.2289	1.0215	3.0082	0.9409	1.6006	**0.2605**
		rho=0.8	17.084	1.663	5.3803	6.3526	1.63639	5.7001	1.4263	2.6162	**0.4683**
	σ=10	rho=0.9	27.365	1.054	8.9108	11.694	2.4838	9.3982	2.1156	4.15	**0.7641**
		rho=0.95	64.736	2.845	21.508	22.618	4.57552	15.342	3.8078	7.8139	**1.6027**
		rho=0.99	335.79	17.68	105.86	100.38	17.4646	29.476	13.798	30.114	**9.1127**
		rho=0	0.013	0.013	0.0129	0.0137	0.01257	0.0129	0.0126	0.0129	**0.0119**
		rho=0.8	0.0249	0.023	0.0243	0.0262	0.02255	0.0243	0.0225	0.0243	**0.0197**
	σ=1	rho=0.9	0.0482	0.038	0.045	0.0509	0.0384	0.045	0.038	0.045	**0.029**
		rho=0.95	0.075	0.05	0.0661	0.0787	0.05164	0.0664	0.0501	0.0661	**0.034**
		rho=0.99	0.346	0.093	0.1817	0.3633	0.12131	0.1955	0.0933	0.1823	**0.0539**
		rho=0	0.3253	0.191	0.2635	0.342	0.19498	0.3225	0.1913	0.2636	**0.0743**
		rho=0.8	0.6235	0.232	0.3619	0.6558	0.26265	0.6067	0.2329	0.3635	**0.0865**
n=250	σ=5	rho=0.9	1.2042	0.302	0.5154	1.2717	0.38471	1.1253	0.306	0.5188	**0.122**
		rho=0.95	1.8746	0.354	0.6845	1.9676	0.48489	1.6599	0.3618	0.6886	**0.1505**
		rho=0.99	8.6511	1.133	2.7125	9.0836	1.63227	4.8871	1.1912	2.7219	**0.5987**
		rho=0	1.3014	0.482	0.751	1.3678	0.51739	1.2898	0.4868	0.7567	**0.1365**
		rho=0.8	2.494	0.639	1.1035	2.6231	0.75954	2.4266	0.6488	1.1072	**0.2072**
	σ=10	rho=0.9	4.8169	0.995	1.963	5.0868	1.25795	4.5012	1.0223	1.9735	**0.3637**
		rho=0.95	7.4984	1.3	2.7296	7.8704	1.68145	6.6395	1.3428	2.7454	**0.4916**
		rho=0.99	34.604	4.566	10.85	36.334	6.31346	19.548	4.8232	10.888	**2.3032**

Source: Simulation results.
Note: the values in bolded form indicate the estimator that has smallest MSE.

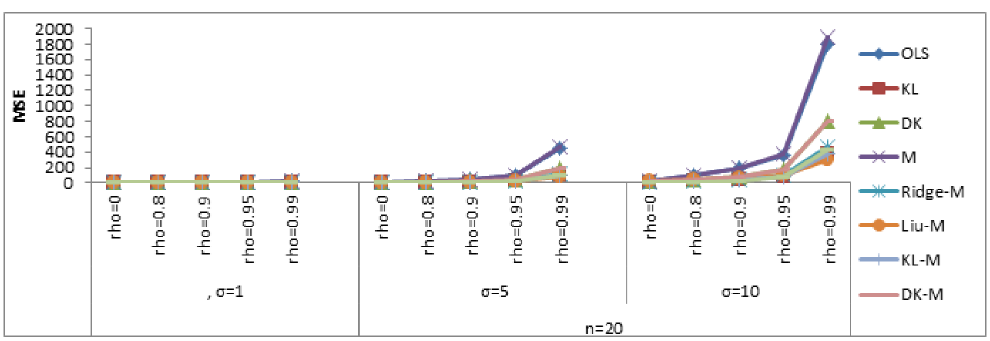

Figure 1. Graph of estimated MSEs of all the estimators when there is no outlier at all levels of multicollinearity and sample size (n) = 20.

4.2. Discussion on simulation results

With respect to the simulation results as displayed in Tables 1–4 and graphically shown in Figures 1–6, the comments are itemized as follows:

(i) As multicollinearity and outliers are simultaneously increasing in the x-direction as expected, OLS performed woefully.
(ii) MSEs of the estimators considered increase as the error variances (σ^2), levels of multicollinearity (rho) and percentage (po), and magnitude (Mo) of outliers increase.

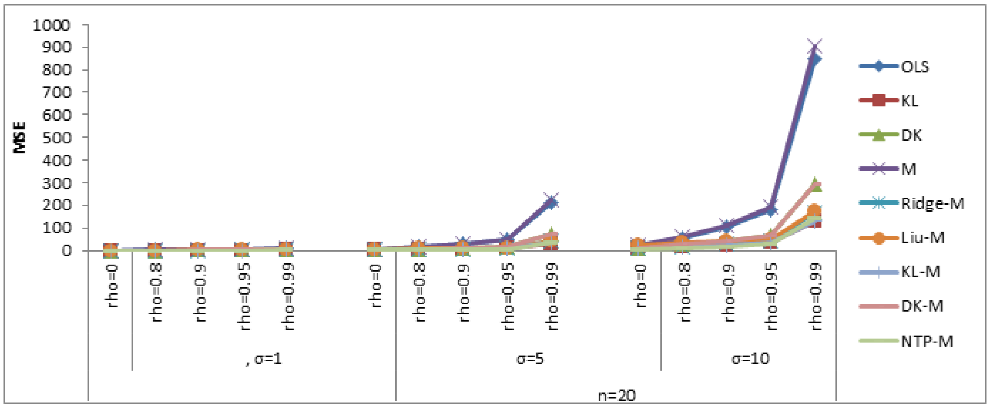

Figure 2. Graph of estimated MSEs of the estimators when the magnitude of outlier is 5, 10%.

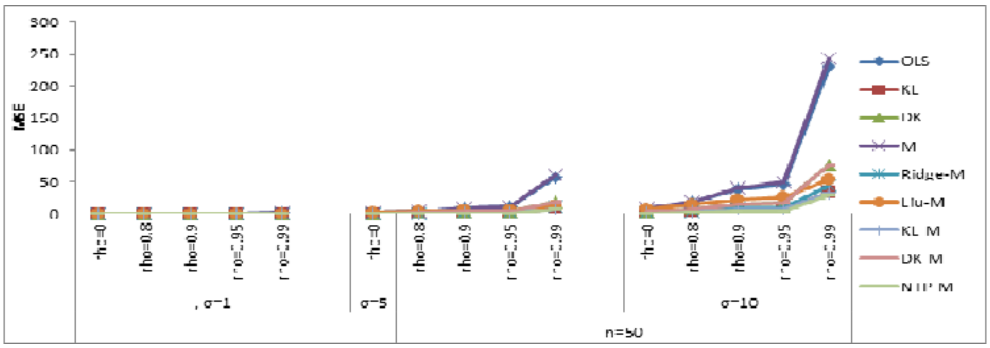

Figure 3. Graph of estimated MSEs of the estimators at 20% outliers when the magnitude of outlier is 5, at all levels of multicollinearity, error variances and $n = 50$.

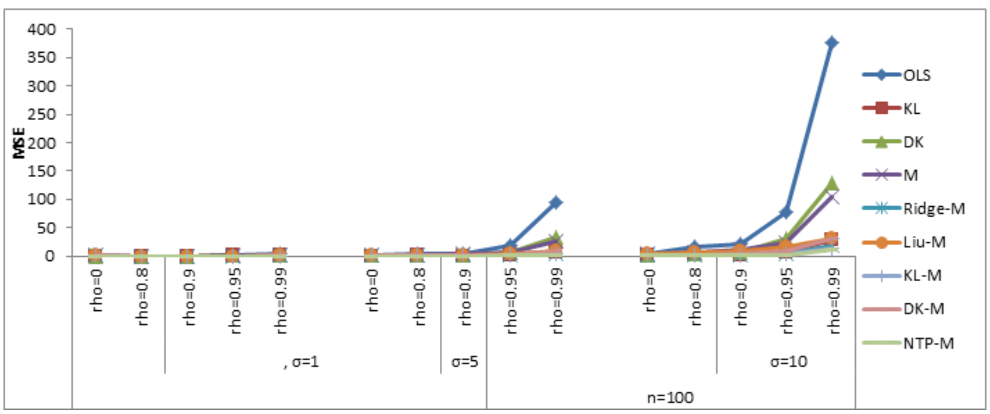

Figure 4. Graph of estimated MSEs of the estimators at 10% outliers when the magnitude of outlier is 10, at all levels of multicollinearity, error variances and $n = 100$.

(iii) As the sample size (n) increases, the MSEs of the estimators decreases.

(iv) When rho>0, Mo>0, the percentage of outliers (po) increases, and sample size (n) increases the RNTP outperformed other estimators considered as the two anomalies occur simultaneously in the x-direction.

4.3. Application to real-life data

Data from Hussein & Abdalla [27] were adopted as real-life application in this study. The linear model below is the regression model for the data set.

$$y = \beta_1 x_2 + \beta_2 x_2 + \beta_3 x_3 , \tag{49}$$

where y is the product value in the manufacturing sector, x_1 is the value of the imported intermediate, x_2 represents the imported capital commodities and x_3 indicates the value of imported raw materials. Ref. [11] claimed that the data suffered from the problem of multicollinearity in the values of the variance inflation factor (VIF), which were estimated to be 128.29, 103.43, and 70.87. Likewise, Ref. [28] affirmed the claim of Ref. [11] and spotted the presence of outliers in the data.

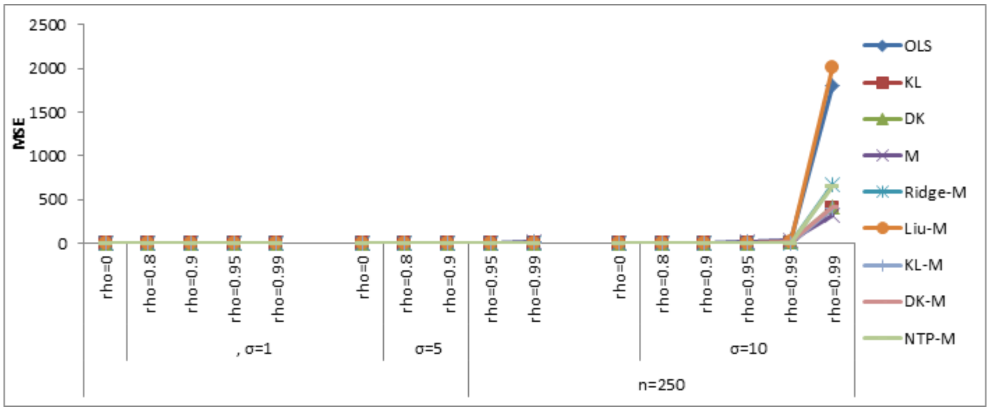

Figure 5. Graph of estimated MSEs of the estimators at 10% outliers when the magnitude of outlier is 5, at all levels of multicollinearity, error variances and $n = 250$.

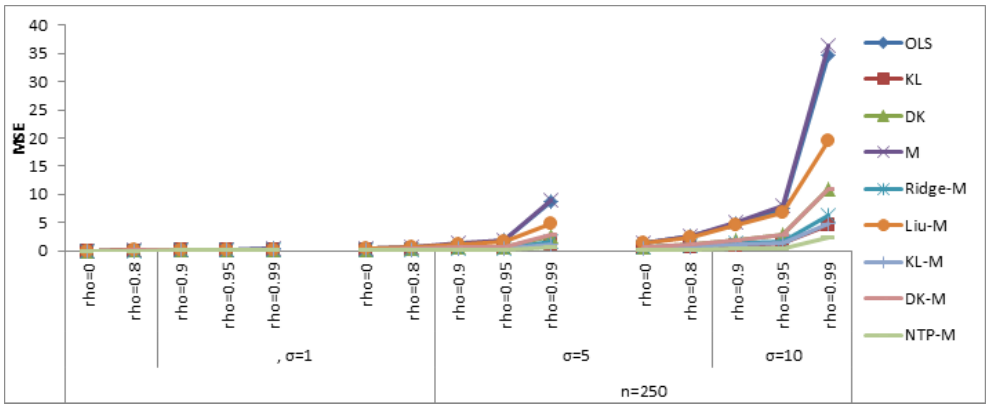

Figure 6. Graph of estimated MSEs of the estimators at 20% outliers when the magnitude of outlier is 20, at all levels of multicollinearity, error variances and $n = 2504$.

Table 5. Regression coefficients and MSEs of estimators using the real-life data.

Coefficients	OLS	KL	DK	NTP	M	RID-M	LIU-M	KL-M	DK-M	NTP-M
β_0	208.885	176.894	200.223	0.1029	173.34	161.419	−4145097	157	194.072	0.11812
β_1	0.61295	0.87701	0.68445	1.7704	0.9976	1.0047	3.6E+07	1.0412	0.73523	1.84258
β_2	1.25626	1.15167	1.22794	0.6457	1.1153	1.10108	7.5E+07	1.0866	1.20783	0.62593
β_3	−1.2213	−1.266	−1.23338	−0.195	−1.1159	−1.2875	1.1E+07	−1.2938	−1.242	−0.2938
MSEs	1850.48	85236.3	1700.39	2.6537	2212.6	1322.55	1.02E+20	103389	1910.32	**2.35401**

Source: R-output

The highlighted value in Table 5 indicates the MSE of the proposed robust estimator which is the least when compared with MSEs of other estimators.

5. Conclusion

When multicollinearity and outliers are present in the data set, ordinary least squares regression analysis remains inconsistent and unreliable. Numerous estimators that can simultaneously address the issues of multicollinearity and outliers have been developed. When the two anomalies in the linear regression model arise, it is still necessary to further investigate another reliable approach and advise the end user of statistics to apply it. This work therefore presented Robust-M New Two-Parameter (RNTP), or NTP-M, and evaluated its performance in comparison to several already existing ones in the presence of multicollinearity and particularly when there are outliers in the x-direction. In order to demonstrate the superiority of the proposed estimator, theoretical expressions under certain circumstances were established. A simulation study was carried out alongside some factors to show that the new robust estimator (RNTP) is better than all other estimators considered in the study. Likewise, real-life data was used to justify the claim.

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