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Mathematical modelling of terrorists re-cycle induced with backward bifurcation

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Abstract

In this work, we formulate a deterministic mathematical model that aims at controlling and preventing terrorism in a population. The model incorporates the prevention of the spread of terrorists in the population by imprisoning individuals caught in the act and allows for natural repentance from terrorism. The main aim of the study is to assess the effect of re-cycling terrorists on the bifurcation phenomena of the model. The model exhibits a backward bifurcation phenomenon. These may have serious implications for the design of intervention programs aiming at eradicating the menace of terrorism. When the values of the scaling factor of the re-cycling rate change from 0 to 1, the bifurcation changes from backward to forward. One or more stable endemic equilibria coexist alongside the stable terrorists-free equilibrium (TFE). It is further demonstrated that the re-cycling of terrorists who have repented from the terrorism act is what triggers the backward bifurcation cycle. This conclusion is important because it shows that eliminating terrorists from the population is no longer dependent on the old precondition of the basic reproduction number being less than one. The parameter of recycling is deleted, and the model is examined, to determine the reason why this model exhibits a backward bifurcation phenomenon. By taking into account the importance of the original model, this was made achievable. When the terrorists aren't recycled, the second model will demonstrate a forward bifurcation, according to an analysis of the second model's coefficient of bifurcation, which is negative.

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1. Introduction

The terrorist attacks on Washington, D.C., New York, and Pennsylvania on September 11, 2001, commonly known as "9/11," were considered acts of war against the entire world. This led to the establishment of an international counter-terrorism effort. According to the United Nations Office of the High Commissioner for Human Rights [1], terrorism involves unlawful attempts or

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threats to kill or seriously injure individuals, along with the destruction of private property, such as government buildings or public spaces. However, the United Nations General Assembly [2] emphasizes, during its 814th and 8496th meetings, that member states have a primary responsibility in preventing and suppressing terrorist acts. Governments are urged to swiftly accede to international counterterrorism treaties and protocols. (Global Terrorism Index [GTI], 2019) [3], stated the absence of a universally recognized definition of terrorism, leading to various competing definitions in the literature. Terrorism is also described as an extreme form of violence used to achieve political goals, forcing innocent people to adopt the ideas of the perpetrators. Over the past five years, there has been a significant decrease in bloodshed and terrorist activities in Syria and Iraq. Iraq experienced the most substantial decline in 2017, and this trend seems to have continued in 2018. The majority of the land and revenue streams once controlled by the Islamic State of Iraq and the Levant (ISIL or ISIS) in Syria and Iraq have been reclaimed. However, affiliate organizations are becoming more active in other regions. A rise in terrorist activities, notably by Al-Qaida, has been observed in the Northern African Maghreb and Sahel regions. More than 9,000 terrorists were present in the area as of March 2018, with a majority located in Libya and Algeria. East Africa is home to the Salafi insurgent organization Al-Shabaab, which seeks to establish an Islamist state in Somalia [3, 4].

Nigeria faced a growing terrorism issue, spreading in the late 1990s [5]. Various Nigerian administrations have implemented policies aligned with counter-terrorism efforts. While casualties related to Boko Haram have decreased due to counter-terrorism operations, violence involving Fulani extremists significantly increased in 2018. Boko Haram, also known as Jamatu Ahlis Sunna Liddaawati wal-Jihad, and Fulani extremists are among Nigeria's deadliest groups. Nigeria ranked third in terms of the number of deaths among the top 10 nations most severely affected by terrorism [6]. Boko Haram and Fulani extremists are the dominant terrorist organizations within Nigeria, with Boko Haram being particularly active in 2022, showing an increase in attacks and casualties compared to the previous year. Boko Haram and Fulani extremists collectively account for 88% of terror-related fatalities and 63% of terrorist incidents in Nigeria. Boko Haram operates primarily in Borno State and has caused significant terrorist activity. The group has disrupted foreign direct investment and humanitarian activities not only in Nigeria but also in neighbouring countries, including Chad, Cameroon, and Niger. Boko Haram split into two factions in 2016, with the more recent Islamic State West Africa Province (ISWAP) pledging allegiance to ISIL. Both organizations consider themselves affiliates of ISIL [7]. Nigeria ranks sixth in terrorism with a terrorism index of 7.86/10 in 2022 [8]. Some investigations encompass various aspects of mathematical modelling and analysis, including calculating epidemic reproduction numbers, conducting stability analysis of the models, exploring global stability in deradicalization models, optimizing resource allocation in counterterrorism operations, and examining human rights considerations in the context of terrorism and counterterrorism efforts [9–14].

Recent studies conducted using mathematical modelling revealed that the spread of terrorism can be compared to an infectious disease [15–19]. In reference [20], the authors split the population into core and non-core groups to represent terrorist and extremist dynamics. Bertozzi *et al*. [21] created a deterministic model for crime and security, while Cooke and Van Den Driessche [10] expanded on this by introducing the spread of radical beliefs, fanaticism, recruitment, and terrorist acts. Despite these models, critical elements like correctional facilities and detailed control strategies were sometimes omitted. Likewise, recent research employing a deterministic model to explore the radicalization process in Kenya, with a specific focus on its influence on rehabilitation institutions but without considering correctional facilities, has been undertaken [22–24]. To combat extremism, the model classified the total population into susceptible, exposed, terrorists, imprisoned, and recovered classes. It should be noted that the model does not account for forced recruitment following the treatment of individuals vulnerable to radicals or recruiters.

Additionally, Bayon *et al.* [25] created a model that included the management of counter-terrorism strategies, dividing it into "fire control" and "water control." Certain elements such as surveillance, incarceration and rehabilitation centres, force, recruitment, and more were not incorporated into the model. Seidl *et al.* [26] made a valuable contribution to the literature by examining optimal control within the framework of a terror queue, highlighting the significance of choosing the most effective strategy. The results of direct military/police involvement were included in reducing terrorism in their dynamic models in references [27, 28]. Deploying the military or the police was found to mitigate the threat of terrorism. Furthermore, the risk of terrorism and political mobilization, and the assignment of security to defend society to emphasize or downplay forecasts was discussed by Koblentz [29].

It's crucial to remember that corrupt police, judges, and jailbreaks have contributed to the re-infiltration of terrorists into the system, sometimes receiving protection from government officials. The objective of this work is to formulate a model incorporating recruitment from terrorist groups and imprisonment, aiming to understand the effect of recycling terrorists within the population.

2. The mathematical model

We formulate a deterministic model for terrorism dynamics with five ordinary differential equations in this paper. We grouped the population into five classes with each class representing a subset of the population. We called the susceptible class(S), the exposed class (E), the terrorists' class (T), the imprisonment class (P), and the recovered class (R). The entire population in this study is considered to remain constants and represents the rate of spontaneously occurring deaths. We denote the recruitment rate by $\wedge$ and force of terrorism by $f = \beta T$ where β represents the rate at which the susceptible individuals are exposed to the terrorist act. Individuals that are exposed become terrorists at the rate ξ, γ, is the rate of repentance from the terrorist act, and the death rate caused by terrorist activities is d. Recovered individuals may acquire a certain level of fear of rejoining terrorism for a certain duration [30] and we assume that repentance individuals are losing that fear at a rate of $(1-\delta)\beta = H$ where $0 < \delta < 1$. The rate at which terrorists

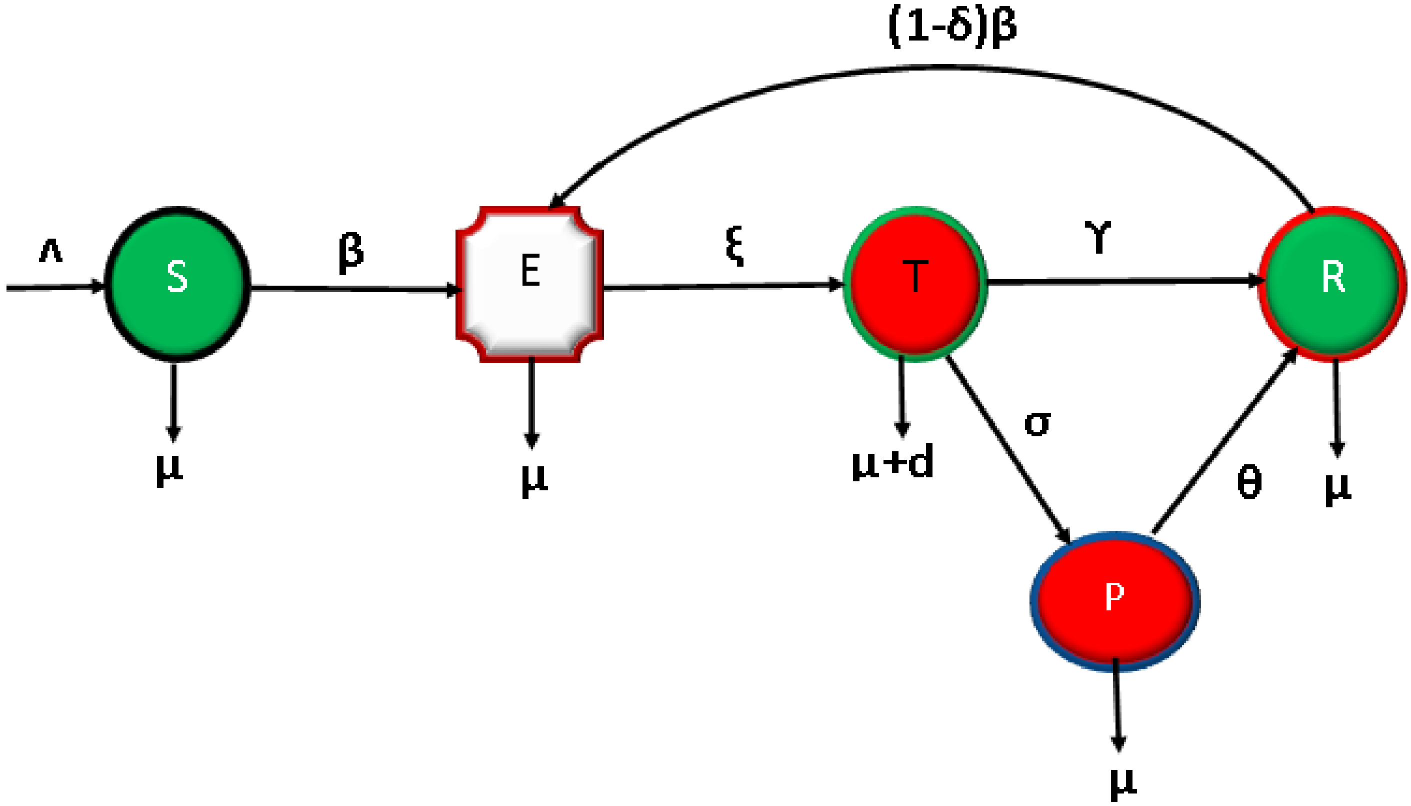

Figure 1: The flowchart of the SETPRE model.

Table 1: Description of the model parameters.

S/No	Parameter	Description
1	Λ	Recruitment rate
2	μ	Natural death
3	d	Death rate due to terrorist activities
4	β	Contact rate
5	ω	Rate of transmission to H
6	σ	Rate at which terrorists are sent to prison
7	δ	Scaling factor for the recycled rate
8	$(1-\delta)\beta = H$	Re-join terrorism rate
9	ξ	Terrorism rate
10	γ	Recovery rate of the terrorist population

are imprisoned is σ and the rate at which they are gotten out of prison is θ. The model diagram is depicted in Figure 1. The system of ordinary differential equations

$$\frac{dS}{dt} = \Lambda - \beta ST - \mu S, \tag{1}$$
$$\frac{dE}{dt} = \beta ST + (1-\delta)\beta TR - (\mu + \xi)E, \tag{2}$$
$$\frac{dT}{dt} = \xi E - (\mu + \gamma + \sigma + d)T, \tag{3}$$
$$\frac{dP}{dt} = \sigma T - (\mu + \theta)P, \tag{4}$$
$$\frac{dR}{dt} = \gamma T + \theta P - (\mu + (1-\delta)\beta T)\, R. \tag{5}$$

Tables 1 and 2 describe the parameters and variables of the SETPRE model respectively.

Table 2: Description of the model variables.

S/No	Variables	Description
1	S	Susceptible subclass
2	E	Exposed subclass
3	T	Terrorist subclass
4	P	Imprison subclass
5	R	Recovered subclass

The initial conditions of the sub-populations are

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, T(0) = T_0 \geq 0, P(0) = P_0 \geq 0, R(0) = R_0 \geq 0. \tag{6}$$

We assumed that all the parameters used in the model are positive for future time.

3. Basic properties of the SETPRE model

We now discuss the properties of the proposed SETPRE model.

3.1. The invariant region

We obtained, in this section, the region where the solution of (1)–(5) is bounded.

Lemma 1. *The initial conditions of the solutions of the system (1)–(5) are contained in the region $\mathbb{R}^5_+$: defined by*

$$B = \left\{ (S, E, T, P, R) \in \mathbb{R}^5_+ : S + E + T + P + R \leq \frac{\Lambda}{\mu} \right\}. \tag{7}$$

The total human population is

$$N = S + E + T + P + R. \tag{8}$$

We differentiate N with respect to time and get

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dT}{dt} + \frac{dP}{dt} + \frac{dR}{dt}. \tag{9}$$

We substituted the values of $\frac{dS}{dt}, \frac{dE}{dt}, \frac{dT}{dt}, \frac{dP}{dt}, \frac{dR}{dt}$ and simplify, we get

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \beta ST - \mu S + \beta ST + (1-\delta)\beta TR - (\mu + \xi)E + \xi E - (\mu + \gamma + \sigma + d)T + \sigma T - (\mu + \theta)P \\ &\quad + \gamma T + \theta P - (\mu + (1-\delta)\beta T) R, \qquad (10) \\ \frac{dN}{dt} &= \Lambda - \mu N - dT. \qquad (11) \end{aligned}$$

If there is no death due to terrorists' activities, then the above equation becomes

$$\frac{dN}{dt} \leq \Lambda - \mu N. \tag{12}$$

Separating the variables, solve for N and apply the initial conditions, we get $0 \leq N \leq \frac{\Lambda}{\mu}$ So, we can conclude that all the solution set of the system (1)–(5) is bounded in B, where

$$B = \left\{ (S, E, T, P, R) \in \mathbb{R}^5_+ : 0 \leq N \leq \frac{\Lambda}{\mu} \right\}. \tag{13}$$

Therefore, if $S(0)$, $E(0)$, $T(0)$, $P(0)$, and $R(0)$ are inside of B, then $S(t)$, $E(t)$, $T(t)$, $P(t)$, and $R(t)$ will stay within B as $t \to \infty$. Similarly, if the initial condition starts outside of B, then the solution will approach B as $t \to \infty$. That completes the proof of the theorem.

3.2. Positivity of the solution

Theorem 1. *Let* $D = \{(S, E, T, P, R) \in \mathbb{R}^5 : S(0) = S_0 > 0, E(0) = E_0 > 0, T(0) = T_0 > 0, P(0) = P_0 > 0, R(0) = R_0 > 0\}$*; then the solutions of* $\{S, E, T, P, R\}$ *are all positive for* $t \geq 0$.

Proof: From equation (1) of the system of differential equations (1)–(5) , we get

$$\frac{dS}{S} \geq -\mu dt. \tag{14}$$

Solving this equation and applying the initial condition yields

$$S(t) \geq S_0 e^{-\mu t} \geq 0. \tag{15}$$

Similarly, from the second, third, fourth, and fifth equations we get

$$E(t) \geq E_0 e^{-(\mu+\xi)t} \geq 0, \tag{16}$$
$$T(t) \geq T_0 e^{-(\mu+\gamma+\sigma+d)t} \geq 0, \tag{17}$$
$$P(t) \geq P_0 e^{-(\mu+\theta)t} \geq 0, \tag{18}$$
$$R(t) \geq R_0 e^{-\mu t} \geq 0. \tag{19}$$

Hence, the proof of the theorem is complete. Therefore, the solution of the model is positive over time ($t \to \infty$).

3.3. Positivity of the solution

The terrorists-free equilibrium (TFE) state, , of the SETPRE model (1)–(5) is obtained by equating the right-hand side of the equation to zero and letting $E = T = P = R = 0$, we solve for S and H and we get:

$$S = \frac{\wedge}{\mu}, \tag{20}$$

$$E_0 = \left(\frac{\wedge}{\mu}, 0, 0, 0, 0\right). \tag{21}$$

3.4. Effective reproduction number R_{ter}

The next-generation matrix method is used to obtain the effective reproduction number [19, 31]. Let F be the matrix of the new terrorist's terms and V the matrix of the transition terms. Then,

$$F = \begin{pmatrix} \beta S T + (1-\delta)\beta T R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

$$V = \begin{pmatrix} (\mu+\xi)E & -\xi E & (\mu+\gamma+\sigma+d)T \\ -\sigma T & (\mu+\theta)P & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Therefore, our next-generation matrices are

$$F = \begin{pmatrix} \frac{df_1}{dE} & \frac{df_1}{dT} & \frac{df_1}{dP} \\ \frac{df_2}{dE} & \frac{df_2}{dT} & \frac{df_2}{dP} \\ \frac{df_3}{dE} & \frac{df_3}{dT} & \frac{df_3}{dP} \end{pmatrix}\Bigg|_{(E_0)} = \begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

and

$$V = \begin{pmatrix} \frac{dv_1}{dE} & \frac{dv_1}{dT} & \frac{dv_1}{dP} \\ \frac{dv_2}{dE} & \frac{dv_2}{dT} & \frac{dv_2}{dP} \\ \frac{dv_3}{dE} & \frac{dv_3}{dT} & \frac{dv_3}{dP} \end{pmatrix}\Bigg|_{(E_0)} = \begin{pmatrix} \mu+\xi & 0 & 0 \\ -\xi & \mu+\gamma+\sigma+d & 0 \\ 0 & -\sigma & \mu+\theta \end{pmatrix}, \tag{25}$$

$$FV^{-1} = \begin{pmatrix} \beta\xi\Lambda/\mu(\mu+\xi)(\gamma+\mu+\sigma+d) & \beta\Lambda/\mu(\gamma+\mu+\sigma+d) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Table 3: Sensitivity indices of the parameters of the basic reproduction numbers.

Parameter	Relationship with other parameters	Indices	Sign
ξ	$\frac{\mu}{\mu+\xi} > 0$	0.095	Positive
β	1	1.000	Positive
Λ	1	1.000	Positive
γ	$\frac{-\gamma}{\gamma+\mu+\sigma+d} < 0$	-0.281	Negative
σ	$\frac{-\sigma}{\gamma+\mu+\sigma+d} < 0$	-0.329	Negative
μ	$\frac{-[(\mu+\xi)(\gamma+\mu+\sigma+d)+\mu(\gamma+\mu+\sigma+d)+\mu(\mu+\xi)]}{(\mu+\xi)(\gamma+\mu+\sigma+d)} < 0$	-2.1208	Negative
d	$\frac{-d}{\gamma+\mu+\sigma+d} < 0$	-0.329	Negative

The characteristic equation of the matrix FV^{-1} is given by

$$\lambda^3 - \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\sigma+d)}\lambda^2. \tag{27}$$

With solutions $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \dfrac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\sigma+d)}$.
Therefore,

$$|FV^{-1} - \lambda I| = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\sigma+d)}. \tag{28}$$

Hence the basic reproduction number is

$$R_{ter} = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\sigma+d)}. \tag{29}$$

3.5. *Sensitivity indices and analysis of the parameters*

We calculate the sensitivity and perform the analysis of the sensitivity of the parameter values using the basic reproductive number to ascertain their effects on the terrorist's activities. As a result, we will be able to determine which control (parameter) causes the biggest reduction in R_{ter} and which control measure is most effective in halting the development of terrorism. To do this, we calculate the reproduction number's normalized forward sensitivity index regarding these variables. This index assesses the correlation between the relative change of the basic reproduction number and changes in the parameter. The formula for the forward normalized sensitivity index of a parameter say *theta* of the reproduction number is given by

$$\Delta_{\theta}^{R_{ter}} = \frac{dR_{ter}}{d\theta} \times \frac{\theta}{R_{ter}}. \tag{30}$$

The results of the computation of the sensitivity indices of parameters are shown in Table 3.

3.5.1. *Interpretation of the sensitivity indices of the parameters*

The parameters in Table 3 show the sensitivity indices of the parameters. The values of $\beta, \wedge, \xi$, have a considerable effect on the spread of terrorists. This is so because as their levels rise, the average number of secondary terrorists also rises. Their values must be reduced to eradicate the terrorists. The other potentially sensitive parameters that are critical for terrorists' control when their levels are raised are the repentance rate γ, the imprisonment rate σ the death rate that occurs naturally μ, and the death rate caused by terrorists' activities d, which have negative sensitivity indices. Figure 2 displays a bar chart depicting the parameters' sensitivity indices. To eradicate terrorists, parameters with their bars pointing upward (those with positive sensitivities indices) must be decreased and those with their bars pointing downward (those with negative sensitivities) must be increased. Although the bars of μ and are point downward, it is not biologically tenable to argue that we should increase the rates of natural death and terrorism-related deaths to eliminate the terrorists because our goal is to prevent life from being lost rather than saved. Figure 2 gives a pictorial representation of the sensitivity indices of the parameters of the basic reproduction number.

3.6. *Local stability analysis of the terrorists-free equilibrium point*

Using the Routh-Hurwitz criteria [23–25], we establish the following theorem.

Theorem 2. *The terrorists-free equilibrium point is locally asymptotically stable if $R_{ter} < 1$ and unstable if $R_{ter} > 1$.*

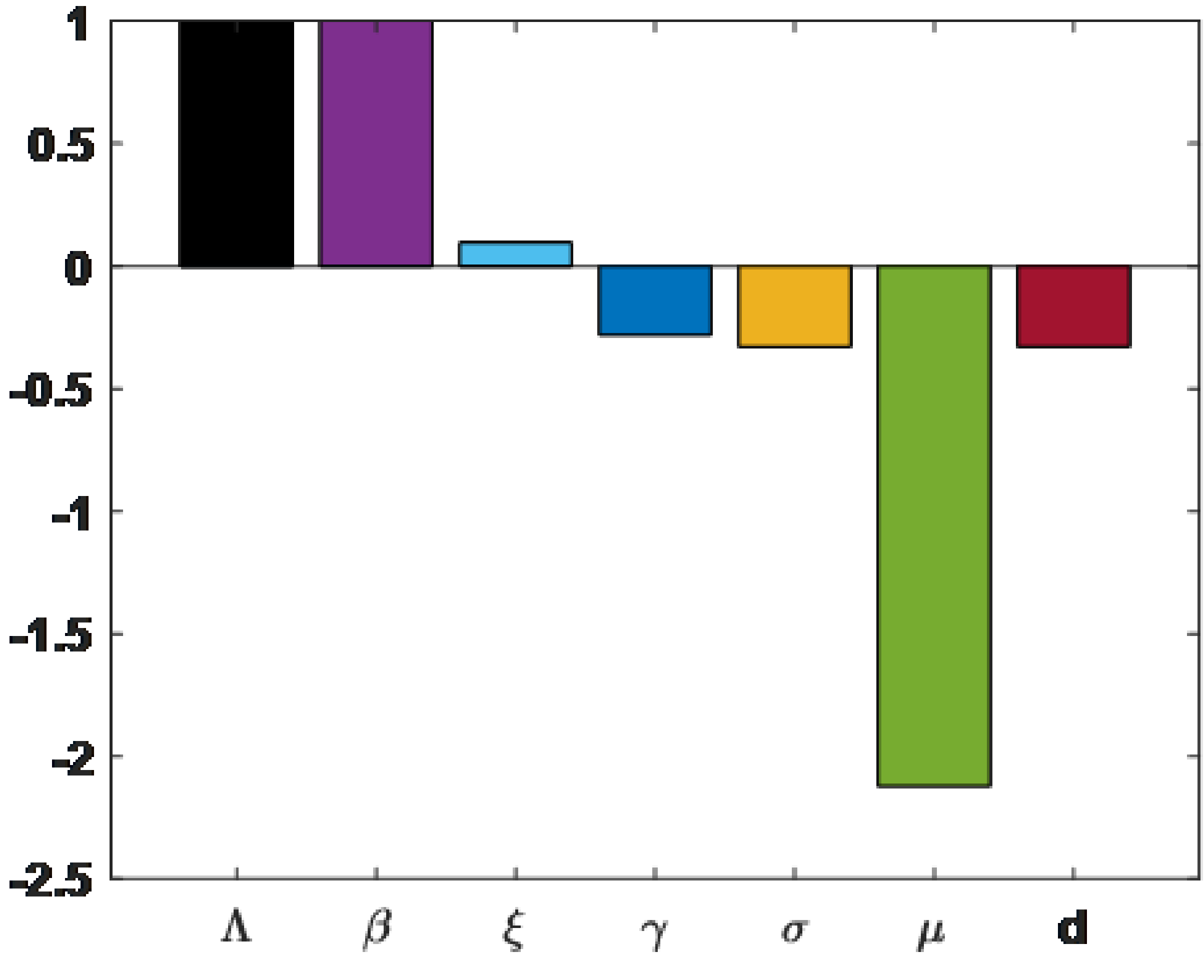

Figure 2: Pictorial representation of the sensitivity indices in a bar chart.

Proof: To prove the local stability of the terrorists' free equilibrium, we obtain the Jacobian matrix of the system (1) at the terrorists' free equilibrium E_0. The Jacobian matrix of the system of equations (1)–(5) at the terrorists' free equilibrium point becomes

$$J(E_0) = \begin{pmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial T} & \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial T} & \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial T} & \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial R} \\ \frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial E} & \frac{\partial f_4}{\partial T} & \frac{\partial f_4}{\partial P} & \frac{\partial f_4}{\partial R} \\ \frac{\partial f_5}{\partial S} & \frac{\partial f_5}{\partial E} & \frac{\partial f_5}{\partial T} & \frac{\partial f_5}{\partial P} & \frac{\partial f_5}{\partial R} \end{pmatrix}\Bigg|_{E_0} = \begin{pmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \xi & -(\mu+\gamma+\sigma+d) & 0 & 0 \\ 0 & 0 & \sigma & -(\mu+\theta) & 0 \\ 0 & 0 & \gamma & \theta & -\mu \end{pmatrix}. \tag{31}$$

With characteristic equation

$$(\lambda+\mu)^2(\lambda+\theta+\mu)\,(\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\sigma+d)-\Lambda\beta\xi)=0, \tag{32}$$

which implies

$$(\lambda+\mu)^2=0, \quad (\lambda+\mu+\theta)=0, \quad \mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\sigma+d)-\Lambda\beta\xi=0. \tag{33}$$

Solving for λ we get

$$\lambda_1 = -\mu, \tag{34}$$
$$\lambda_2 = -\mu, \tag{35}$$
$$\lambda_3 = -\mu-\theta. \tag{36}$$

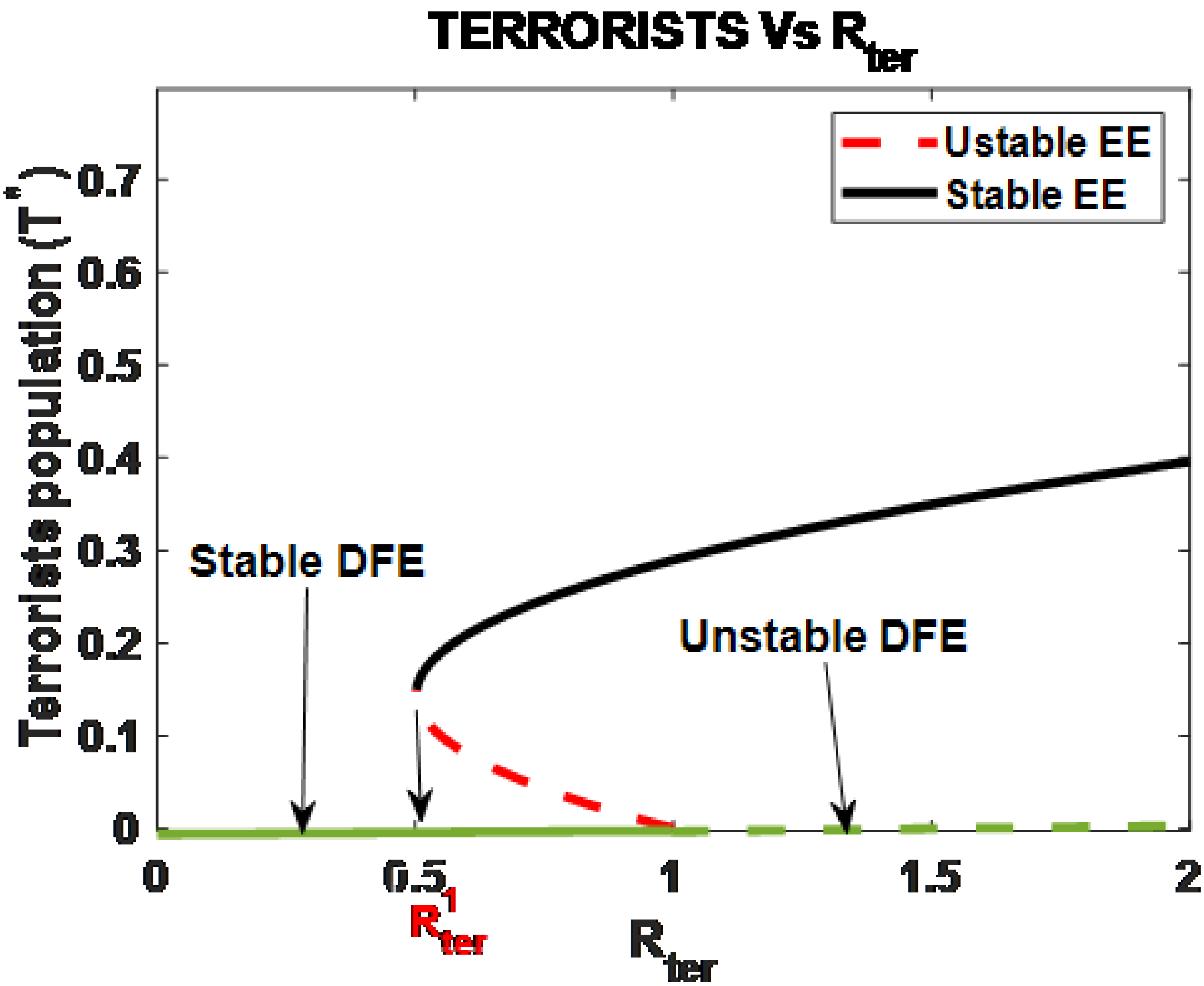

Figure 3: Backward bifurcation diagram of terrorists' population versus basic reproduction number.

The roots of the quadratic equation below are the other two eigenvalues,

$$\mu\lambda^2 + \mu(d + \sigma + \gamma + 2\mu + \xi)\lambda + \mu(\mu + \xi)(\gamma + \mu + \sigma + d) - \Lambda\beta\xi = 0, \tag{37}$$

which simplifies to

$$\mu\lambda^2 + \mu(d + \sigma + \gamma + 2\mu + \xi)\lambda + \mu(\mu + \xi)(\gamma + \mu + \sigma + d)(1 - R_{ter}) = 0. \tag{38}$$

This can be written as

$$M_0\lambda^2 + M_1\lambda + M_2 = 0, \tag{39}$$

where

$$M_0 = \mu, \tag{40}$$
$$M_1 = \mu(d + \sigma + \gamma + 2\mu + \xi), \tag{41}$$
$$M_2 = \mu(\mu + \xi)(\gamma + \mu + \sigma + d)(1 - R_{ter}). \tag{42}$$

Using Routh Hurwitz criteria, the eigenvalues of the matrix all have negative real parts, and so the system of equation with the above characteristic equation is locally asymptotically stable.

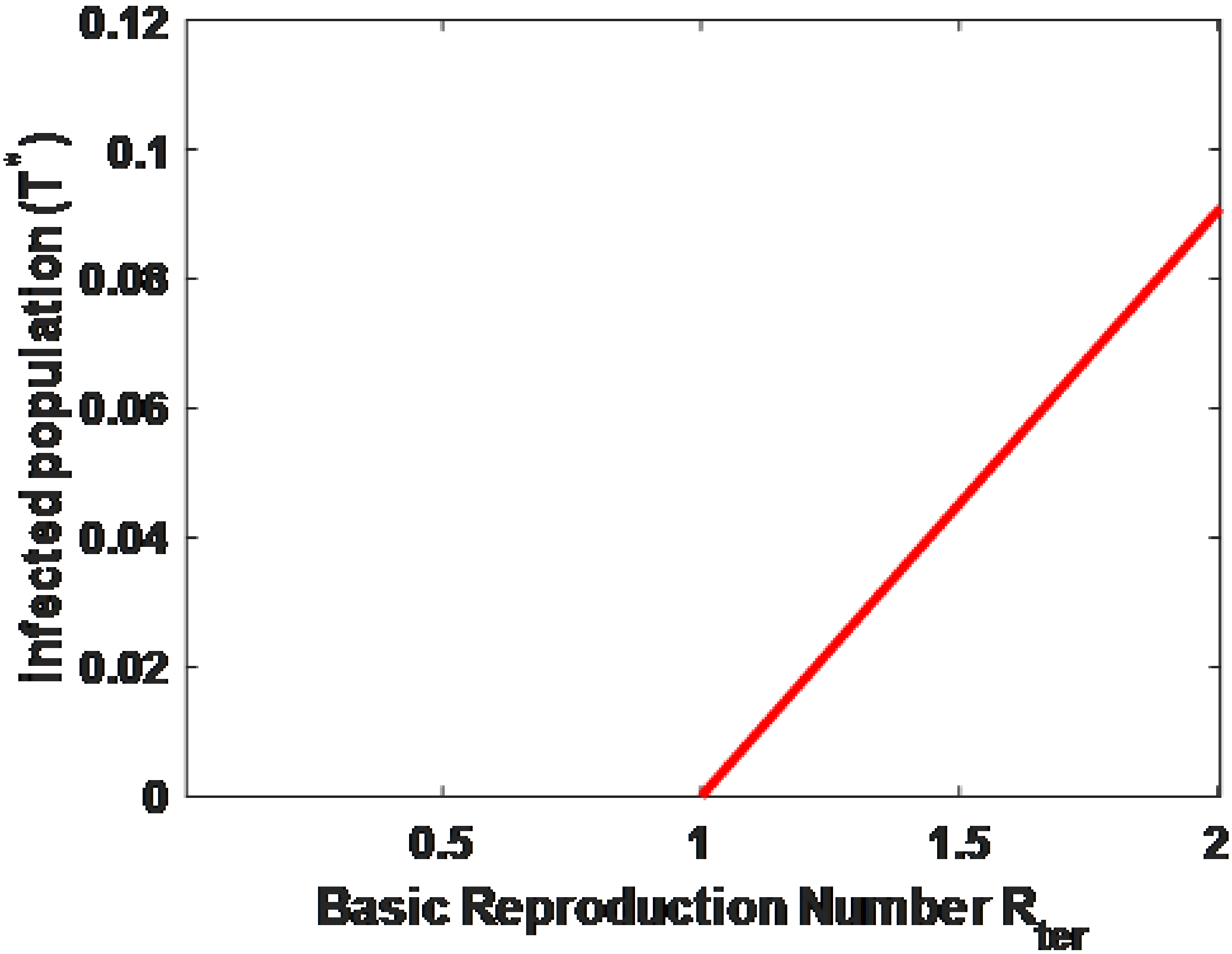

Figure 4: Forward bifurcation when the terrorists' population is plotted against the basic reproduction number

3.7. Global stability analysis of the terrorists-free equilibrium point

We shall investigate the global stability of the system using the techniques implemented by Castillo-Chavez and Song. Let's re-write the system (1)–(5) as follows:

$$\frac{dX}{dt} = F(X, Z), \tag{43}$$

$$\frac{dZ}{dt} = G(X, Z), \quad G(X, 0) = 0. \tag{44}$$

where X represents the terrorists' free population, that is $X = \{S, R\}$ while Z represents the terrorists' population, that is $Z = \{E, T, P\}$. The terrorists' free equilibrium point of the model is represented by $U = (X^*, 0)$.

The point $U = (X^*, 0)$ is a globally asymptotically stable equilibrium for the model provided that $R_{ter} < 1$, which is locally asymptotically stable, and the following conditions must be met:
(H_1): For $\frac{dX}{dt} = F(X, 0)$, X^* is globally asymptotically stable.
(H_2): $G(X, Z) = AZ - \tilde{G}(X, Z)$, $\tilde{G}(X, Z) \geq 0$ for $(X, Z) \in \Omega$.

If the model (1)–(5) meets the given two criteria, then the following theorem holds.

Theorem 3. *The terrorists' free equilibrium point* $U = (X^*, 0)$ *is globally asymptotically stable provided* $R_{ter} < 1$ *and conditions* (H_1) *and* (H_2) *are satisfied.*

Proof: From the system, we can get $F(X,Z)$ and $G(X,Z)$:

$$F(X,Z) = \begin{pmatrix} \Lambda - \beta S T - \mu S \\ \gamma T + \theta P - (\mu + (1-\delta)\beta T)R \end{pmatrix}, \tag{45}$$

$$G(X,Z) = \begin{pmatrix} \beta S T + (1-\delta)\beta T R - (\mu + \xi)E \\ \xi E - (\mu + \gamma + \sigma + d)T \\ \sigma T - (\mu + \theta)P \end{pmatrix}. \tag{46}$$

At $E = T = P = R = 0$,

$$\frac{dX}{dt} = F(X,0) = \begin{pmatrix} \Lambda - \mu S \\ 0 \end{pmatrix}. \tag{47}$$

From the above system, we see that $X^* = (\Lambda/\mu, 0)$ is a globally asymptotic point. This can be verified from the solutions, namely

$$S(t) = \frac{\Lambda}{\mu} + \left(S(0) - \frac{\Lambda}{\mu} \right) e^{-\mu t}. \tag{48}$$

As $t \to \infty$, the solution $S(\infty) \to \Lambda/\mu$. This implies the global convergence of (2) in Ω and this satisfies condition H_1. Now from H_2 we have that $G(X,Z) = AZ - \tilde{G}(X,Z), \tilde{G}(X,Z) \geq 0$ for $(X,Z) \in \Omega$. Therefore, $\tilde{G}(X,Z) = AZ - G(X,Z)$. Where A is an $n \times n$ matrix, Z is a column vector, and $G(X,Z)$ is a column vector formed from the corruption equations.

The first partial derivative of $G(X,Z)$ concerning E, T, and P, computed at the terrorists-free equilibrium point gives matrix A.

$$A = \begin{pmatrix} -(\mu+\xi) & \dfrac{\beta\Lambda}{\mu} & 0 \\ \xi & -(\mu+\gamma+\sigma+d) & 0 \\ 0 & \sigma & -(\mu+\theta) \end{pmatrix}, \tag{49}$$

$$AZ = \begin{pmatrix} -(\mu+\xi)E + \dfrac{\beta\Lambda}{\mu}T \\ \xi E - (\mu+\gamma+\sigma+d)T \\ \sigma T - (\mu+\theta)P \end{pmatrix}. \tag{50}$$

From the expression $\tilde{G}(X,Z) = AZ - G(X,Z)$, we have

$$\tilde{G}(X,Z) = \begin{pmatrix} -(\mu+\xi)E + \dfrac{\beta\Lambda}{\mu}T \\ -(\mu+\gamma+\sigma+d)T \\ -(\mu+\theta)P \end{pmatrix} - \begin{pmatrix} \beta S T + (1-\delta)\beta T R - (\mu+\xi)E \\ \xi E - (\mu+\gamma+\sigma+d)T \\ \sigma T - (\mu+\theta)P \end{pmatrix}, \tag{51}$$

$$\tilde{G}(X,Z) = \begin{pmatrix} -(\mu+\xi)E + (\delta-1)\beta R \\ 0 \\ 0 \end{pmatrix}. \tag{52}$$

From $\tilde{G}(X,Z)$, we can see that $\tilde{G}_1(X,Z) \leq 0$, $\tilde{G}_2(X,Z) = 0$, which leads to $\tilde{G}(X,Z) \leq 0$. That means the second condition (H_2) is not satisfied. Thus, the system of equations may not be globally asymptotically stable when $R_{ter} < 1$.

3.8. Interior equilibrium point (IEP)

We compute the Equilibrium for the Equilibrium Point (IEP) of the system (1)–(5) in terms of the terrorists (T) using the model system equation (1)–(5) , which gives

$$E_* = \left(\frac{\Lambda}{\beta T^* + \mu}, \frac{(\gamma+\mu+\sigma+d)T}{\xi}, T^*, \frac{\sigma T}{(\mu+\theta)}, \frac{\gamma(\mu+\theta)T + \theta\sigma T}{(\mu+\theta)(\mu+(1-\delta)\beta T)} \right). \tag{53}$$

4. Determination of the direction of bifurcation

To explore the possibility of the backward or forward bifurcation of the model system (1)–(5) we use the Center manifold theory. The backward or forward bifurcation phenomena will help us determine the local stability of the system at the endemic equilibrium point. Now from the basic reproduction number. We compute the basic reproductive number as follows:

$$R_{ter} = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\sigma+d)}. \tag{54}$$

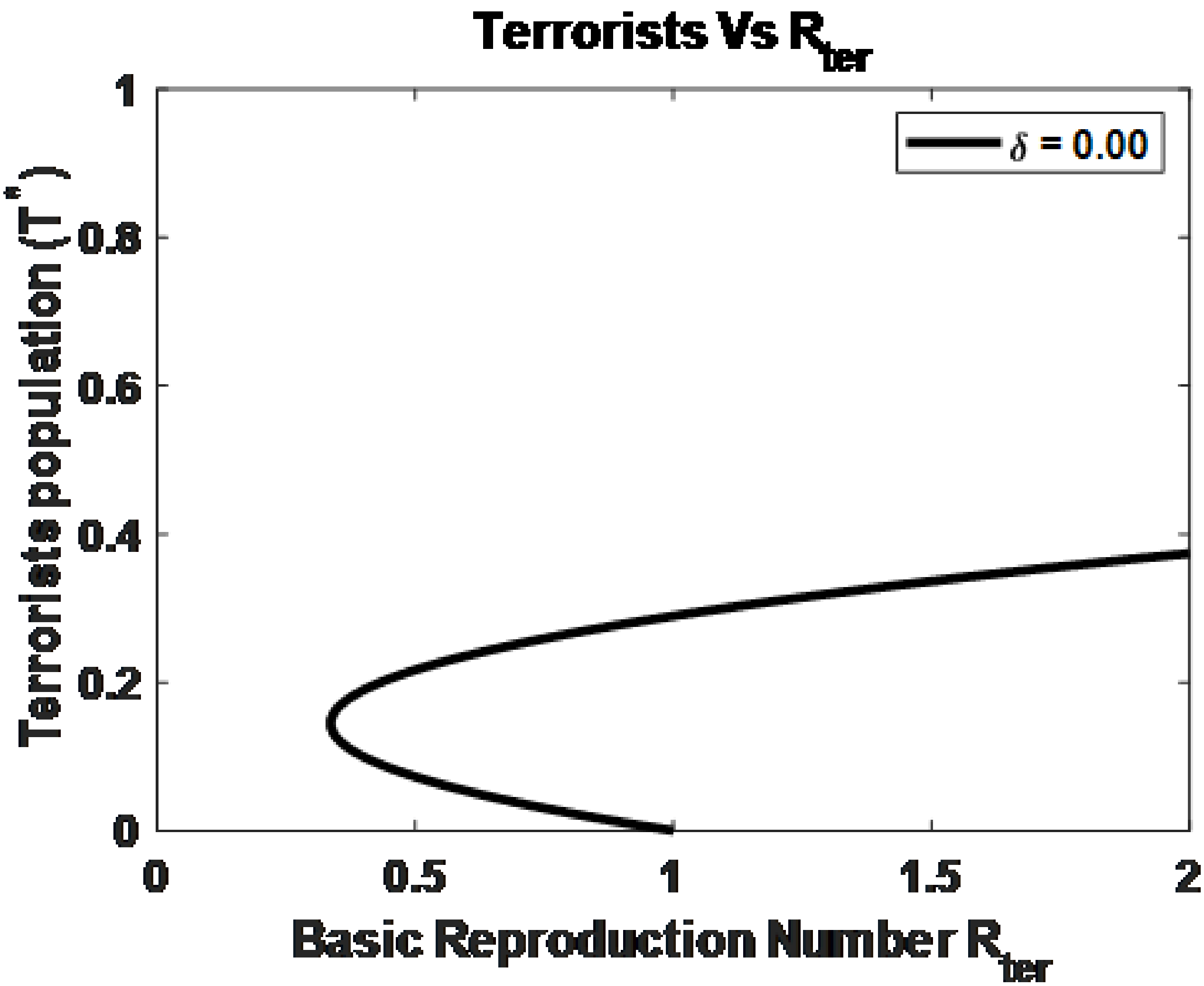

Figure 5: The bifurcation diagram when $\delta = 0$

Let's set $\beta = \beta^*$ as the bifurcation parameter and consider the value of R_{cor} at 1. We make β the subject of the formula to obtain:

$$\beta^* = \frac{\mu(\mu + \xi)(\gamma + \mu + \sigma + d)}{\xi \Lambda}. \tag{55}$$

Let's rename the variables as $S = x_1$, $E = x_2$, $T = x_3$, $P = x_4$, and $R = x_5$. Also, let's use the vector $X = (x_1, x_2, x_3, x_4, x_5)^T$ formulated as $\frac{dX}{dt} = F(x)$, where $F = (f_1, f_2, f_3, f_4, f_5)^T$. The terrorists-free equilibrium becomes $X_0 = (x_1 = \frac{\pi}{\mu}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0)$. Substituting these values of $S = x_1, E = x_2, T = x_3, P = x_4$, and $R = x_5$ into our model system of equations, we get:

$$\frac{dx_1}{dt} = \pi - \beta x_3 x_1 - \mu x_1, \tag{56}$$

$$\frac{dx_2}{dt} = \beta x_3 x_1 + (1 - \delta)\beta x_3 x_5 - (\mu + \xi)x_2, \tag{57}$$

$$\frac{dx_3}{dt} = \xi x_2 - (\gamma + \mu + \sigma + d)x_3, \tag{58}$$

$$\frac{dx_4}{dt} = \sigma x_3 - (\mu + \theta)x_4, \tag{59}$$

$$\frac{dx_5}{dt} = \gamma x_3 + \theta x_4 - (\mu + (1 - \delta)\beta x_3)x_5. \tag{60}$$

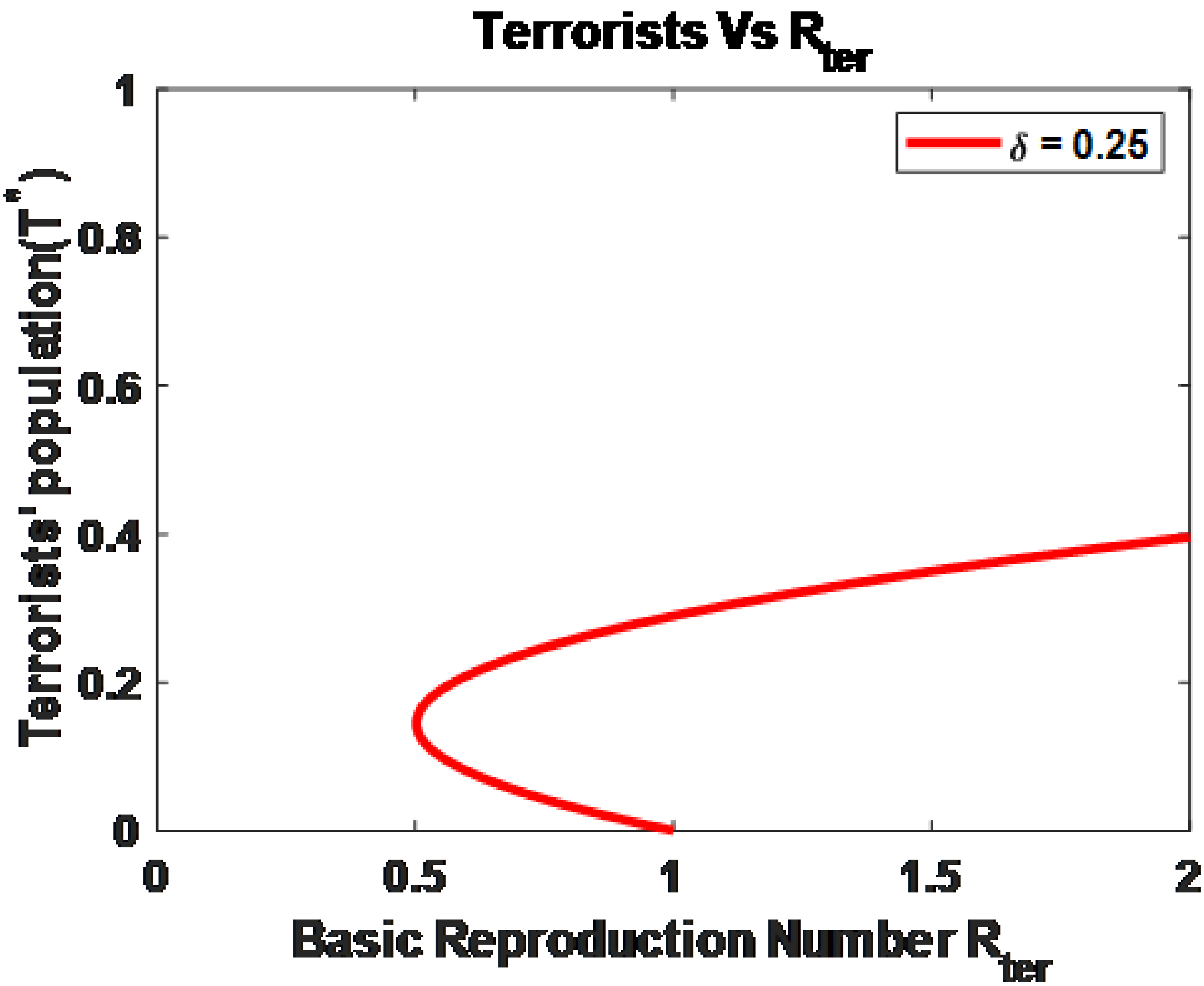

Figure 6: The bifurcation diagram when $\delta = 0.25$

and its Jacobian at the terrorist-free equilibrium point is:

$$J(E^0_{ter}) = \begin{bmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \xi & -(\mu+\gamma+\sigma+d) & 0 & 0 \\ 0 & 0 & \sigma & -(\mu+\theta) & 0 \\ 0 & 0 & \gamma & \theta & -\mu \end{bmatrix}. \tag{61}$$

with the characteristic equation below:

$$(\lambda+\mu)^2(\lambda+\theta+\mu)(\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\sigma+d)-\Lambda\beta\xi) = 0. \tag{62}$$

This means that

$$(\lambda+\mu)^2 = 0, \tag{63}$$

$$(\lambda+\mu+\theta) = 0, \tag{64}$$

$$\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\sigma+d)-\Lambda\beta\xi = 0. \tag{65}$$

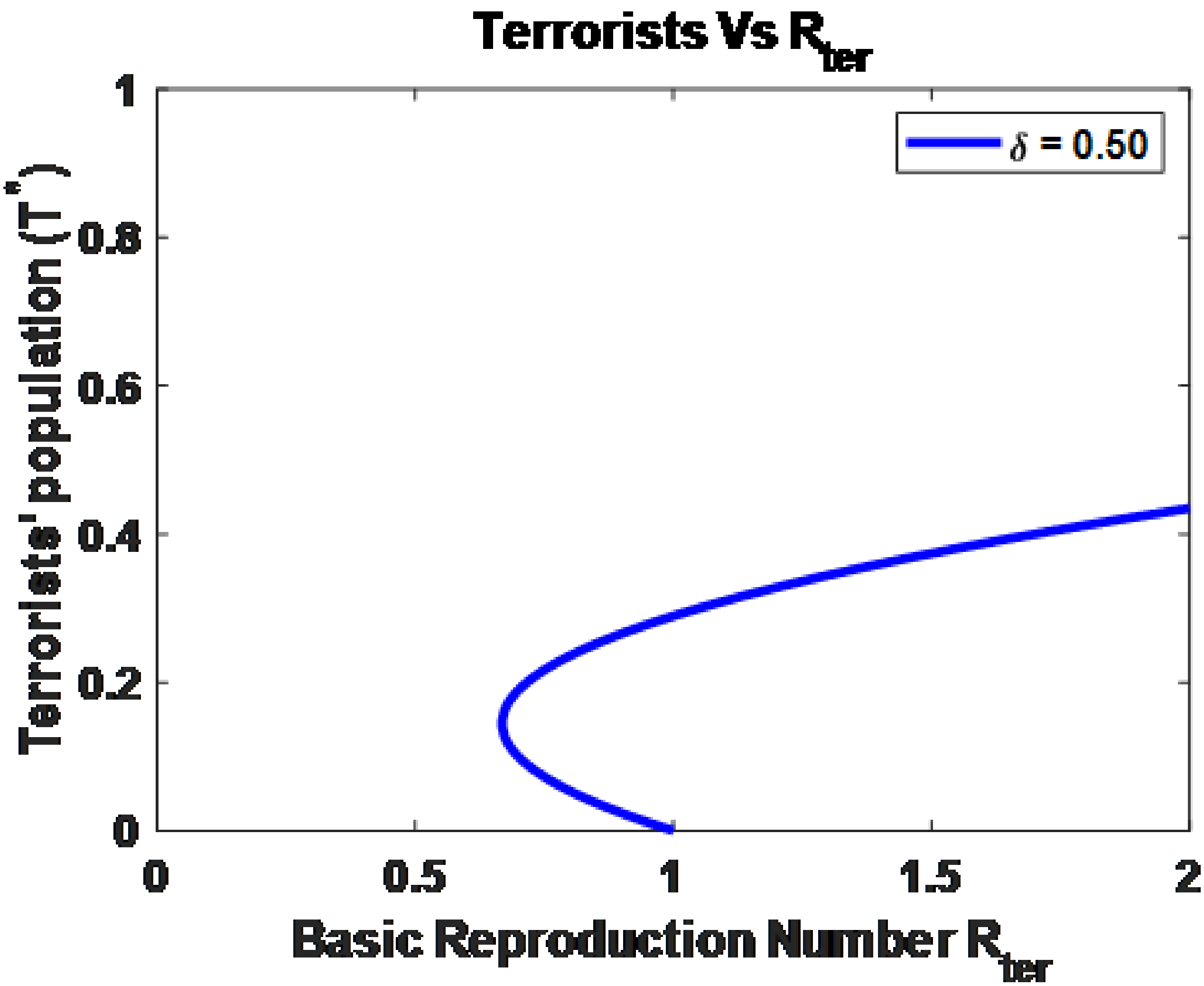

Figure 7: The bifurcation diagram when $\delta = 0.50$

Solving for λ, we get:

$$\lambda_1 = -\mu, \tag{66}$$
$$\lambda_2 = -\mu, \tag{67}$$
$$\lambda_3 = -(\mu + \theta). \tag{68}$$

The remaining two eigenvalues are the roots of the quadratic equation:

$$\mu\lambda^2 + \mu(d + \sigma + \gamma + 2\mu + \xi)\lambda + \mu(\mu + \xi)(\gamma + \mu + \sigma + d) - \Lambda\beta\xi = 0. \tag{69}$$

This can be simplified to:

$$\mu\lambda^2 + \mu(d + \sigma + \gamma + 2\mu + \xi)\lambda + \mu(\mu + \xi)(\gamma + \mu + \sigma + d)(1 - R_{ter}) = 0. \tag{70}$$

Substituting the value of β, we get:

$$\mu\lambda^2 + \mu(d + \sigma + \gamma + 2\mu + \xi)\lambda + \mu(\mu + \xi)(\gamma + \mu + \sigma + d)(1 - 1) = 0, \tag{71}$$
$$\mu\lambda^2 + \mu(d + \sigma + \gamma + 2\mu + \xi)\lambda = 0, \tag{72}$$
$$\lambda(\lambda + \mu(d + \sigma + \gamma + 2\mu + \xi)) = 0, \tag{73}$$
$$\lambda_4 = 0, \quad \lambda_5 = -\mu(d + \sigma + \gamma + 2\mu + \xi). \tag{74}$$

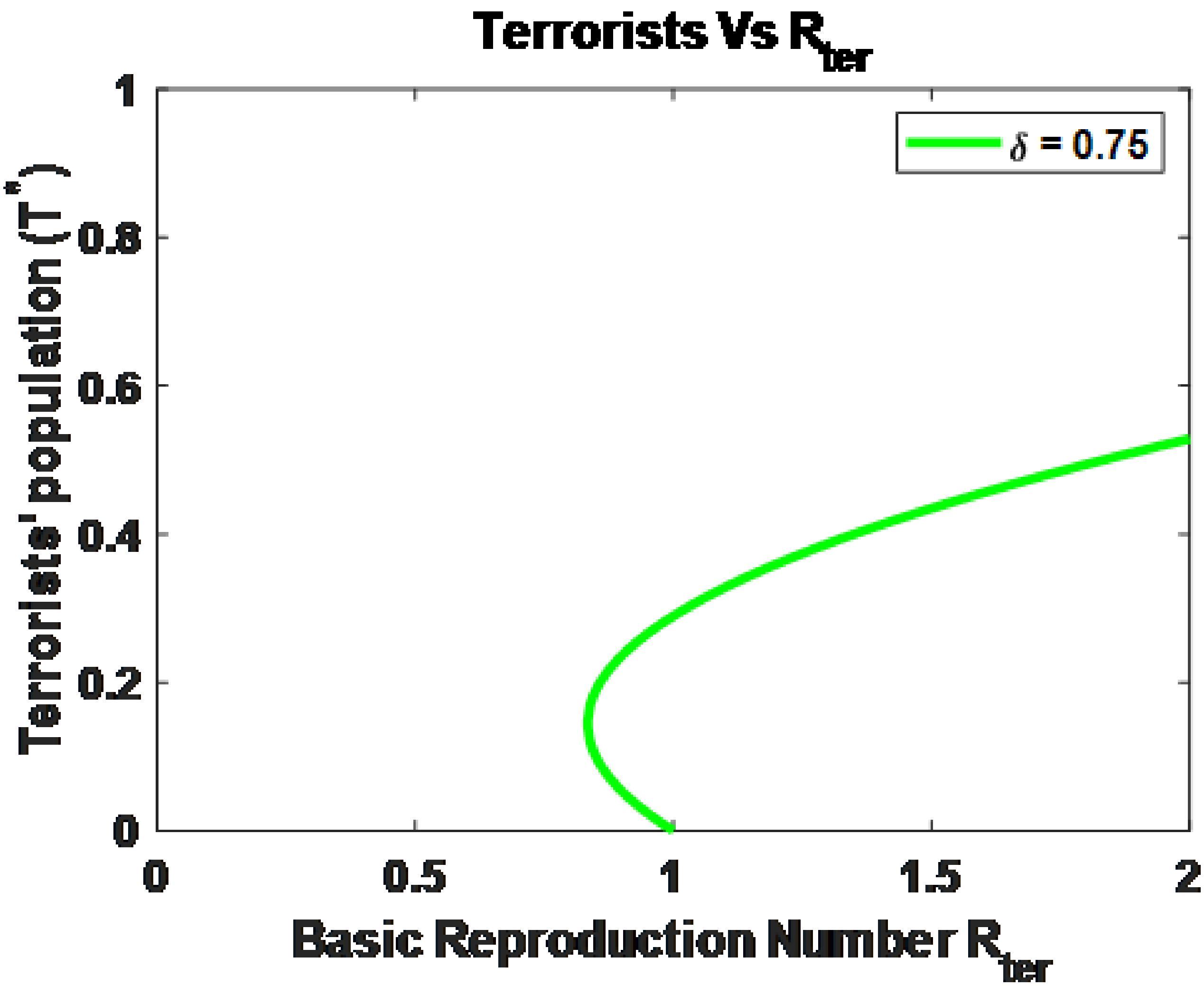

Figure 8: The bifurcation diagram when $\delta = 0.75$

Since λ_3 is a simple eigenvalue, we can use the center manifold theorem. So, we proceed with the computation as follows:

$$W = (w_1, w_2, w_3,\ w_4,\ w_5)^T,$$

$$\begin{bmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \xi & -(\mu+\gamma+\sigma+d) & 0 & 0 \\ 0 & 0 & \sigma & -(\mu+\theta) & 0 \\ 0 & 0 & \gamma & \theta & -\mu \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{75}$$

$$W = \begin{bmatrix} -\frac{\beta\lambda W_3}{\mu^2} & \frac{\beta\Lambda W_3}{\mu(\mu+\xi)} & W_3 & \frac{\sigma W_3}{\mu+\theta} & \frac{\gamma(\mu+\theta)W_3+\sigma\theta W_3}{\mu(\mu+\theta)} \end{bmatrix}^T, \tag{76}$$

$$\begin{bmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\xi) & \xi & 0 & 0 \\ -\frac{\beta\Lambda}{\mu} & \frac{\beta\Lambda}{\mu} & -(\mu+\gamma+\sigma+d) & \sigma & \gamma \\ 0 & 0 & 0 & -(\mu+\theta) & \theta \\ 0 & 0 & 0 & 0 & -\mu \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{77}$$

$$V = \begin{bmatrix} 0 & \frac{\xi V_3}{\mu+\xi} & V_3 & 0 & 0 \end{bmatrix}^T. \tag{78}$$

We now use center manifold theorem as stated in Castillo-Chavez and Song.

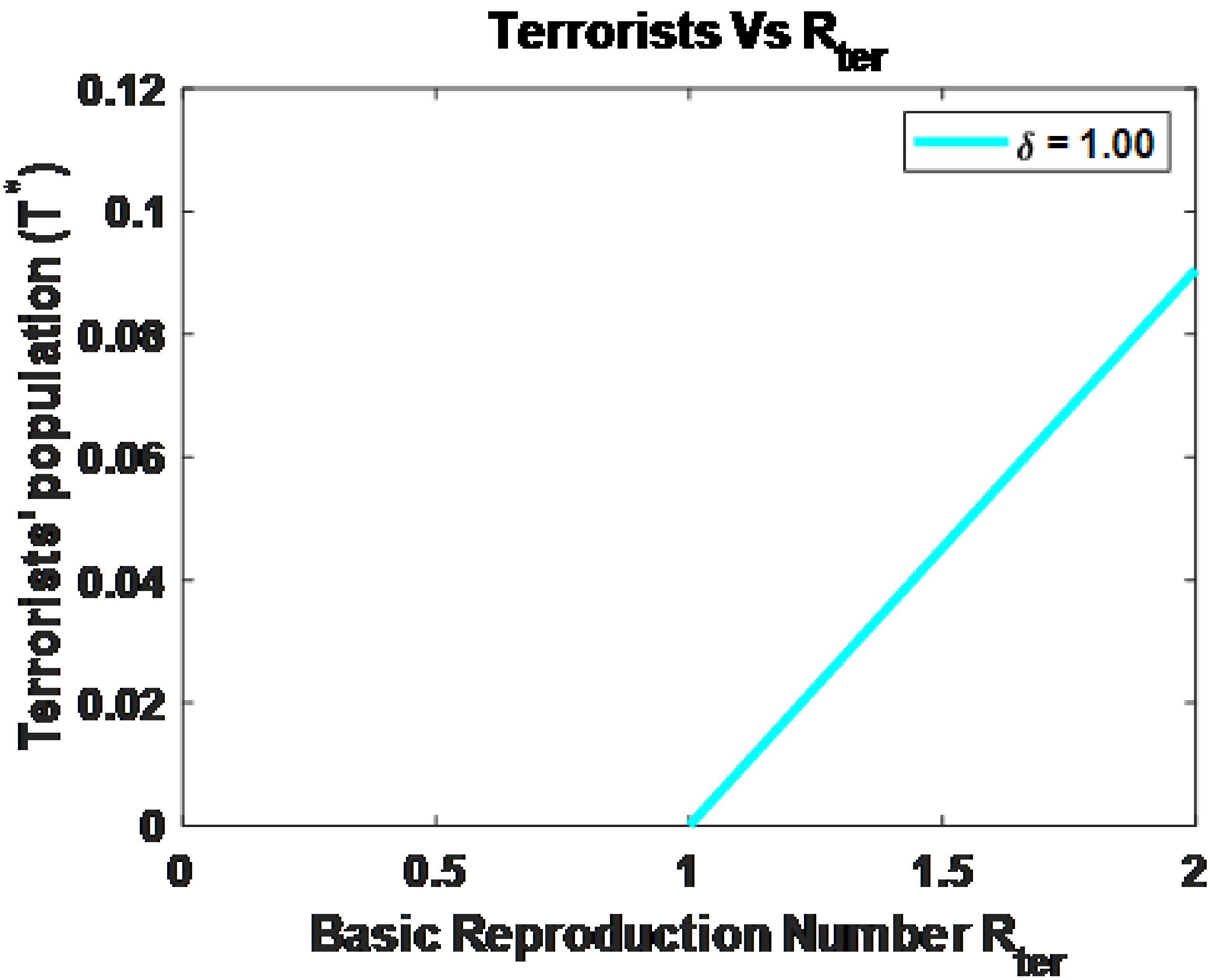

Figure 9: The bifurcation diagram when $\delta = 1.0$

Theorem 4. *Consider the following general system of ordinary differential equation with parameter* φ
$\frac{dx}{dt} = f(x, \varphi), f : R^n xR and f \in C^2(R^n xR)$,
where 0 is an equilibrium point of the system (that is $f(x, \varphi) \equiv 0$*) for all* φ *assume:*

1. $\mathcal{A} = D_x f(0, 0) = (\frac{df_i}{dx_j}(0, 0))$ *is the linearization matrix of the system around the equilibrium point 0 with* φ *evaluated at 0;*

2. *Zero is a simple eigenvalue of* $\mathcal{A}$ *and all other eigenvalues of* $\mathcal{A}$ *have negative real parts*

3. *Matrix* $\mathcal{A}$ *has right eigenvector W and a left eigenvector V corresponding to the zero eigenvalue.*

Let f_k *be the* K^{th} *component of* f *and*

$$\mathcal{A} = \sum_{k,i,j=0}^{n} V_k W_i W_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_0),$$

$$\mathcal{B} = \sum_{k,i=0}^{n} V_k W_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(E_0, \beta).$$

Then the local dynamics of the system around the x=0 are totally determined by $\mathcal{A}$ *and* $\mathcal{B}$*. Particularly,*

1. $\mathcal{A} > 0, \mathcal{B} > 0$ *when* $\varphi < 0$ *with* $||\varphi|| \ll 1, (0, 0)$ *is locally asymptotically stable and there exists a positive unstable equilibrium; when ,* $0 < \varphi \ll 1, (0, 0)$ *is unstable there exists a negative and locally asymptotically stable equilibrium.*

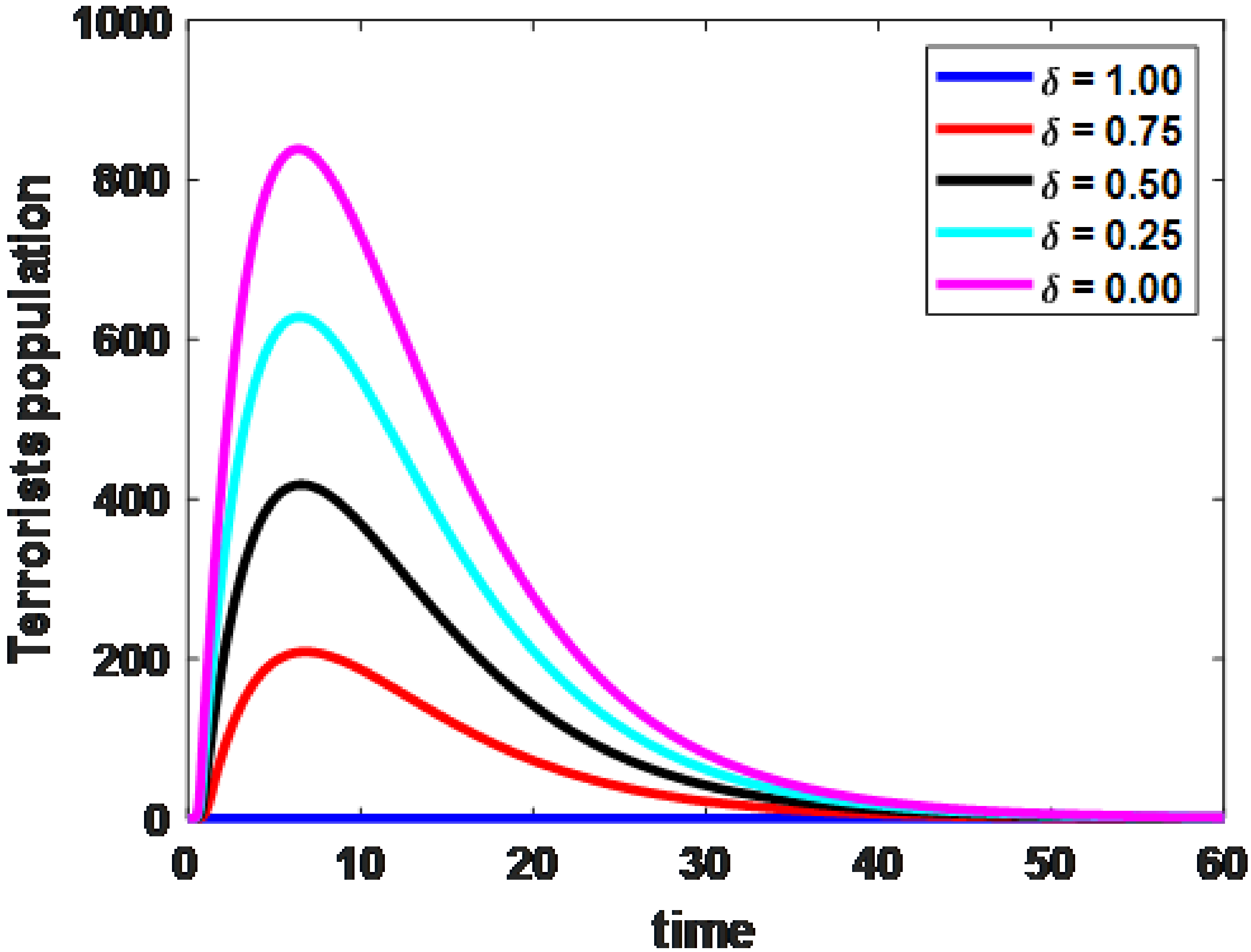

Figure 10: The curve of Terrorists population as δ varies

2. $\mathcal{A} < 0, \mathcal{B} < 0$ *when* $\varphi < 0$,*with* $||\varphi|| \ll 1, (0,0)$ *is locally unstable and there exists a positive unstable equilibrium.*

3. $\mathcal{A} > 0, \mathcal{B} < 0$ *with,* $\varphi < 0$ *is unstable, and there exists locally asymptotically stable equilibrium; when ,* $||\varphi|| \ll 1, (0,0)$ *is stable and positive unstable equilibrium appears*

4. $\mathcal{A} < 0, \mathcal{B} > 0$ *when* φ *changes from negative to positive, x=0 changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes locally asymptotically stable.*

After some calculation and simplification, we got the value of A as

$$A = \frac{2\beta\xi V_3 W_3^2}{\mu^2(\mu+\xi)(\mu+\theta)} Q, \tag{79}$$

$$\text{where} \quad Q = \mu(\gamma(\mu+\theta)+\sigma\theta)(1-\delta) - \beta\Lambda(\mu+\theta). \tag{80}$$

Substituting the values of the eigenvectors and derivatives and simplified, we got B to be

$$B = \frac{\xi\beta\Lambda}{\mu(\mu+\xi)} V_3 W_3 > 0. \tag{81}$$

We have forward bifurcation if

$$\mu(\gamma(\mu+\theta)+\sigma\theta)(1-\delta) - \beta\Lambda(\mu+\theta) < 0 \quad \text{or} \quad \mu(\gamma(\mu+\theta)+\sigma\theta)(1-\delta) < \beta\Lambda(\mu+\theta), \tag{82}$$

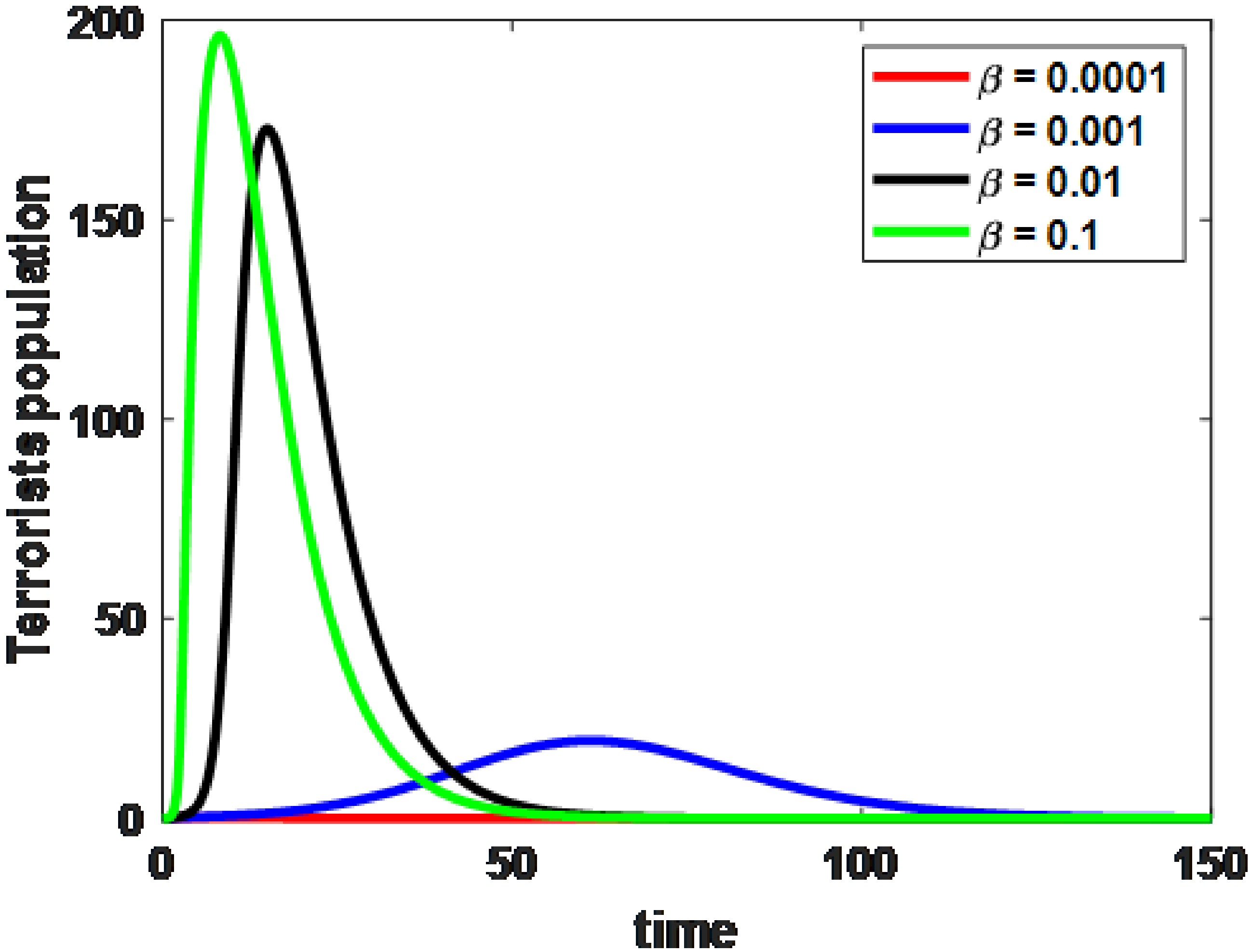

Figure 11: The curve of Terrorists population as β varies

and backward bifurcation if

$$\mu(\gamma(\mu+\theta)+\sigma\theta)(1-\delta)-\beta\Lambda(\mu+\theta)>0 \quad \text{or} \quad \mu(\gamma(\mu+\theta)+\sigma\theta)(1-\delta)>\beta\Lambda(\mu+\theta). \tag{83}$$

4.1. Analysis and plot of the bifurcation diagram

When the values of S, E, T, P, and R are substituted into equation (2) and simplified, we get

$$B_0T^2+B_1T+B_2=0, \tag{84}$$

where

$$B_0=\mu\beta^2(\delta-1)(\sigma(\theta+\mu+\xi)+(\theta+\mu)(\gamma+\mu+\xi)), \tag{85}$$

$$B_1=-(\beta\mu-\beta\mu(\delta-1))(\theta+\mu)(\mu+\xi)(\sigma+\gamma+\mu+d),$$
$$-\Lambda\beta^2\xi(\delta-1)(\theta+\mu)-\beta\mu\xi(\gamma(\theta+\mu)+\sigma\theta)(\delta-1), \tag{86}$$

$$B_2=-\mu^2(\theta+\mu)(\mu+\xi)(\sigma+\gamma+\mu+d)(1-R_{ter}). \tag{87}$$

We plot the value of T^* against R_{ter} to get the bifurcation diagram in Figure 3. The term ‘backward bifurcation’ typically refers to a phenomenon in mathematical models where the dynamics of a system change, leading to the coexistence of multiple stable states. In the context of terrorism, this might imply that there is a critical value of R0, often less than 1, where terrorism dynamics exhibit complex behaviour. Below this critical threshold, counterterrorism efforts might be relatively effective, but beyond it, controlling terrorism becomes significantly more challenging.

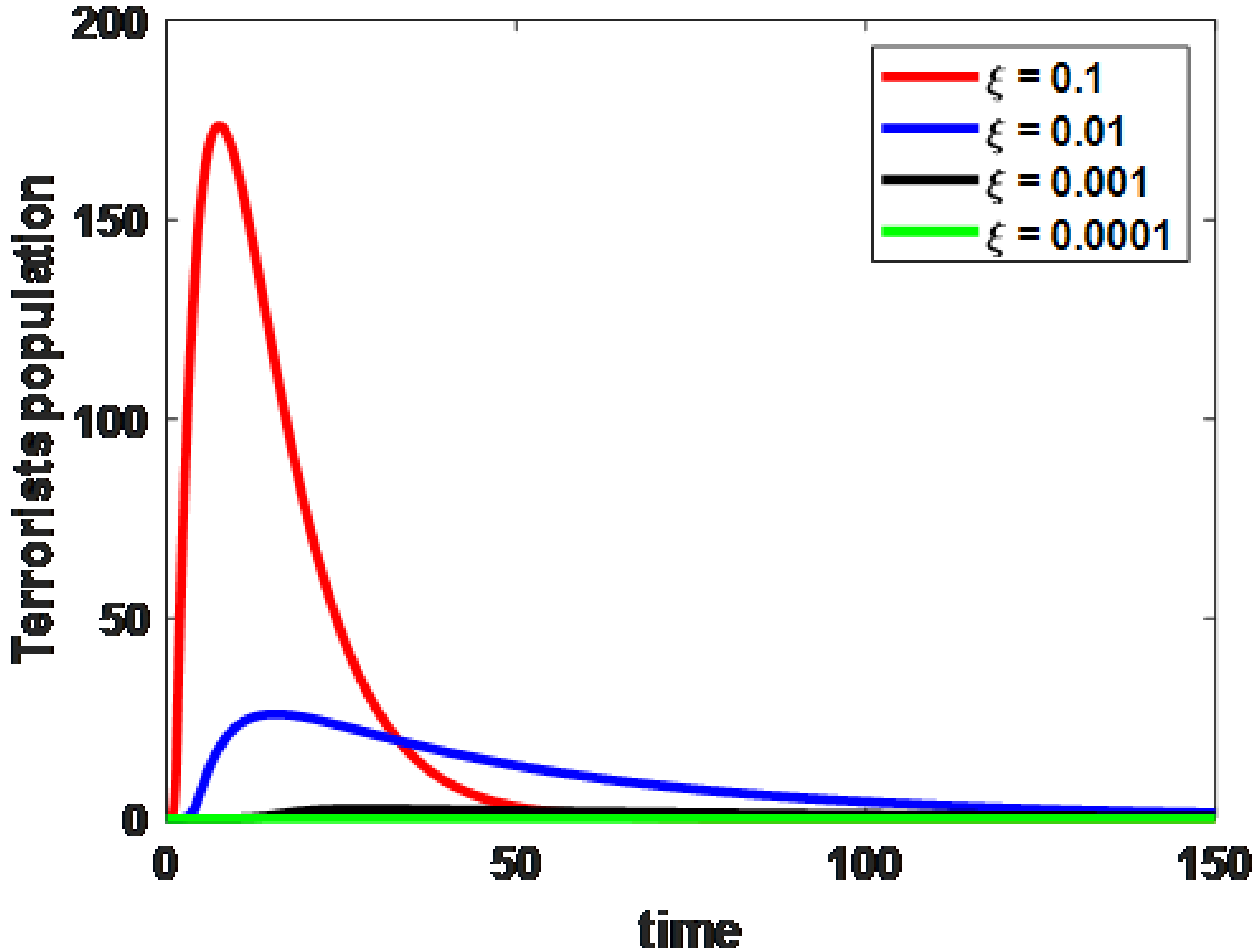

Figure 12: The curve of Terrorists population as ξ varies

4.2. *Effect of terrorists re-cycling on bifurcation of the model*

Let $(1-\delta)\beta = \text{H}$ be the terrorists' recycling rate then, differentiating equation(78) with respect to H, we have

$$\frac{\partial A}{\partial H} = \frac{2\beta\xi\mu V_3 W_3^2}{\mu^2(\mu+\xi)(\mu+\theta)}(\gamma(\mu+\theta)+\sigma\theta). \tag{88}$$

The sign of the bifurcation coefficient A is positive. Hence, the model exhibits phenomena of the backward bifurcation. Therefore, factors that allow terrorists to be recycled into the population should be dealt with to eradicate the menace of terrorism in our society.

4.3. *The model without terrorism re-cycling ($\delta = 1$)*

When there is no recycling of terrorists, the value of A becomes

$$A = -\frac{2\beta\xi V_3 W_3^2}{\mu^2(\mu+\xi)(\mu+\theta)}(\beta\Lambda(\mu+\theta)). \tag{89}$$

This shows that $A < 0$ and $B > 0$, and the equation (2) becomes

$$g(T) = B_1 T + B_2 = 0, \tag{90}$$

where

$$B_1 = \beta\mu(\theta+\mu)(\mu+\xi)(\sigma+\gamma+\mu+d), \tag{91}$$

$$B_2 = \mu^2(\theta+\mu)(\mu+\xi)(\sigma+\gamma+\mu+d)(1-R_{\text{ter}}). \tag{92}$$

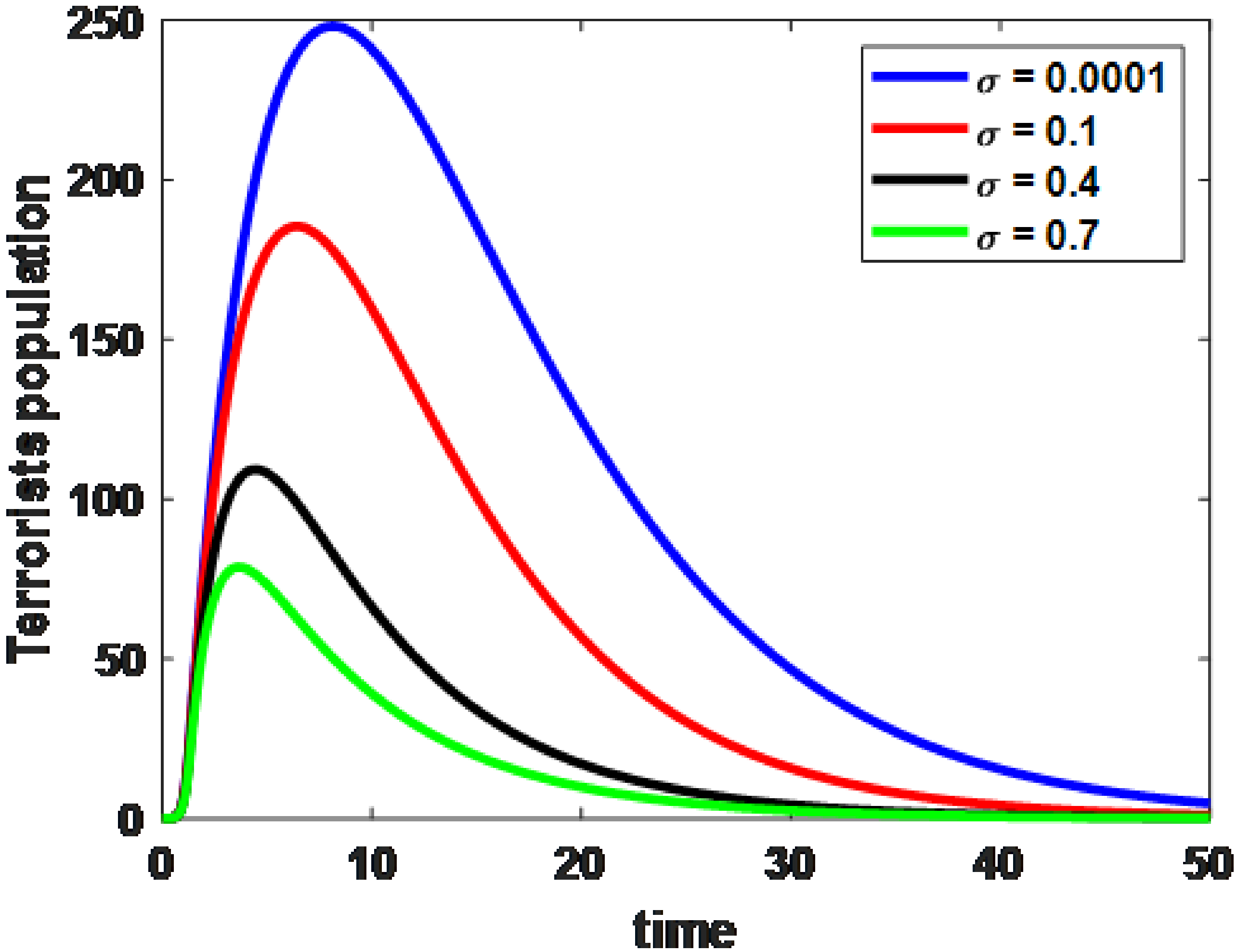

Figure 13: The curve of Terrorists population as σ varies

This is an equation of a straight line with gradient B_1 and T-intercept B_2. The graph of the model gives a forward bifurcation as depicted in Figure 4.

4.4. The model without terrorism imprisonment ($\sigma = 0$)

The value of A becomes

$$A = \frac{2\beta\xi V_3 W_3^2}{\mu(\mu+\xi)(\mu+\theta)}[\gamma\mu(1-\delta) - \beta\Lambda], \tag{93}$$

and the derivative of A with respect to σ is

$$\frac{\partial A}{\partial \sigma} = \frac{2\theta\beta\xi V_3 W_3^2}{\mu^2(\mu+\xi)(\mu+\theta)}(1-\delta). \tag{94}$$

This shows that the sign of the bifurcation coefficient A is positive, indicating that there will be backward bifurcation. This demonstrates that whether terrorists are put in correctional centres or not, the sign of the value of the bifurcation parameter depends on the recycling of terrorists. It can be easily seen that when there is no recycling of terrorists, i.e., $\delta = 1$, then the sign of the values of the bifurcation coefficient A is negative, indicating forward bifurcation (see Figure 4).

4.5. Global stability analysis of the the interior (positive) equilibrium point

Theorem 5. *: If $R_{ter} > 1$, the interior equilibrium E^0_{cov} of the model is globally asymptotically stable.*

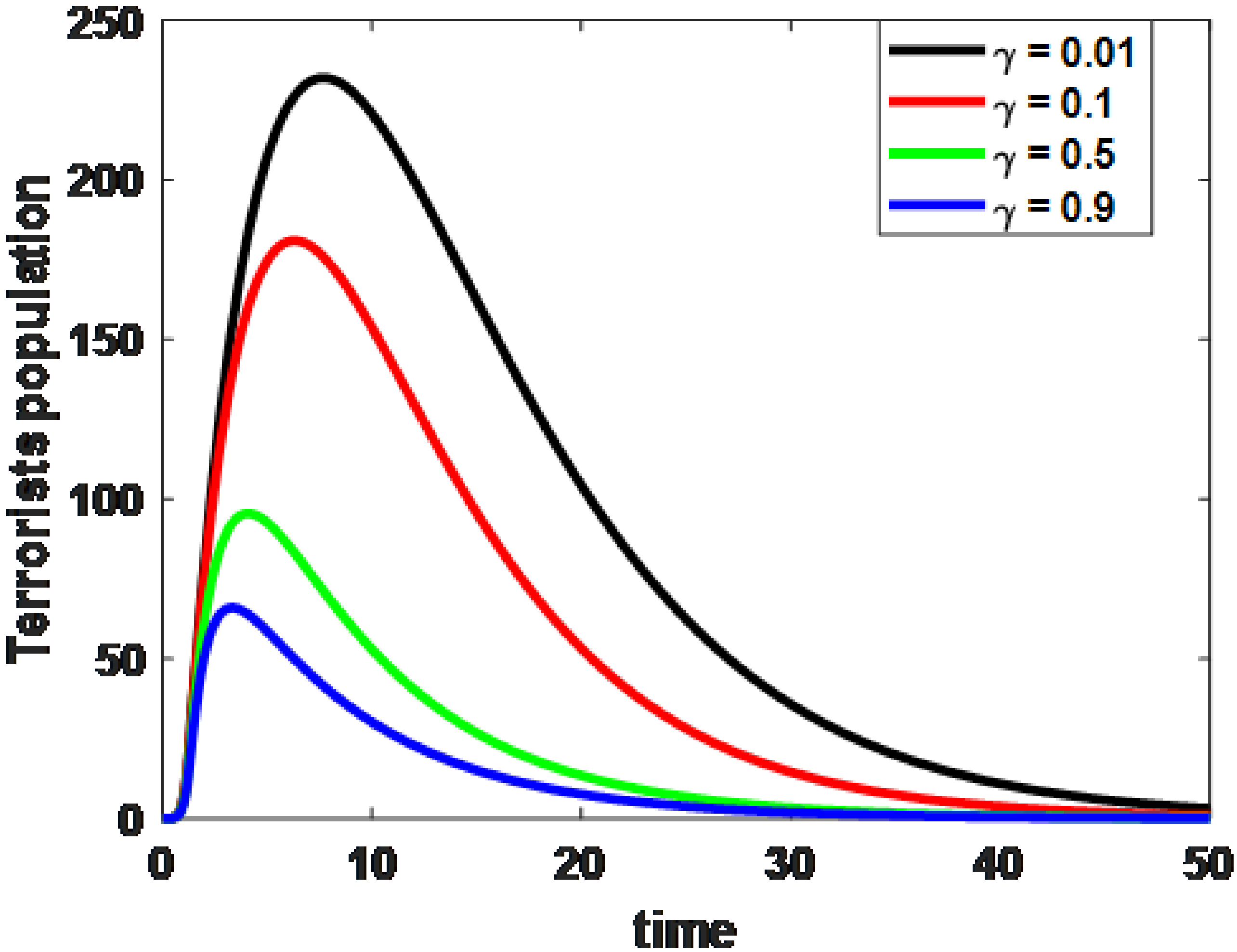

Figure 14: The curve of Terrorists population as γ varies

Proof: Using Lyapunov's direct method and LaSalle's Invariant principle, we prove the above theorem by defining a Lyapunov's function

$$L(S^*, E^*, I^*, R^*) = (S - S^* - \ln\frac{S^*}{S}) + (E - E^* - E^*\ln\frac{E^*}{E}) + (I - I^* - I^*\ln\frac{I^*}{I}) + (P - P^* - P^*\ln\frac{P^*}{P}) + (R - R^* - R^*\ln\frac{R^*}{R}). \quad (95)$$

Differentiating L with respect to t produces

$$\frac{dL}{dt} = \frac{S - S^*}{S}\frac{dS}{dt} + \frac{E - E^*}{E}\frac{dE}{dt} + \frac{I - I^*}{I}\frac{dI}{dt} + \frac{P - P^*}{P}\frac{dP}{dt} + \frac{R - R^*}{R}\frac{dR}{dt}. \quad (96)$$

Substituting the values of $\frac{dS}{dt}$, $\frac{dE}{dt}$, $\frac{dI}{dt}$, and $\frac{dR}{dt}$ into $\frac{dL}{dt}$ and then simplifying, we get

$$\frac{dL}{dt} = Q_1 - Q_2, \quad (97)$$

where

$$Q_1 = \Lambda + \frac{S^*}{S}\beta ST + S^*\mu + E^*(\mu + \xi) + \xi E + T^*(\mu + \gamma + \sigma + d) + P^*(\mu + \theta), \quad (98)$$

$$+ R^*(\mu + (1 - \delta)\beta T), \quad (99)$$

$$Q_2 = \frac{S^*}{S}\Lambda + \mu S + (\mu + \xi)E + \frac{E^*}{E}\beta ST + \frac{E^*}{E}(1 - \delta)\beta TR, \quad (100)$$

$$+ \mu T + dT + \frac{T^*}{T}\xi E + \mu P + \frac{P^*}{P}\sigma T + \mu R + \frac{R^*}{R}\gamma T + \frac{R^*}{R}\theta P. \quad (101)$$

Table 4: Parameter values and sources

S/No	Parameter	Value
1	β	0.50
2	Λ	0.00157
3	μ	0.013077
4	γ	0.06
5	d	0.07
6	δ	(0 - 1) (assumed)
7	θ	0.56
8	ξ	0.12502
9	σ	0.07 (assumed)

$\frac{dL}{dt} \leq 0$ if Q_1 is less than Q_2, $\frac{dL}{dt} = 0$ if and only if $S = S^*$, $E = E^*$, $I = I^*$, $P = P^*$, $R = R^*$. Therefore, the largest invariant set in $\{(S^*, E^*, I^*, P^*, R^*) \in \Omega : \frac{dL}{dt} = 0\}$ is the singleton set E_0^*, where E_0^* is the interior equilibrium of the system (1)–(5). By LaSalle's Invariant principle, this implies that E_{cov}^* is globally asymptotically stable in Ω if Q_1 is less than Q_2.

5. Numerical simulation and discussion of the result

We used the values of the parameters in Table 4 to show how the terrorism re-cycling parameter affects the bifurcation phenomenon of the model. Refer to references [30–32] for the estimation of parameters. Figures 5–9 show the direction of the bifurcation as we vary the value of δ from 0 to 1 while Figures 10–14 show the variation in the population of the terrorists as $\delta, \beta, \xi, \sigma$ and γ are varied respectively.

In this work, we formulated a mathematical model for terrorism's transmission. Our model considered recovered individuals going back to terrorist activities after some time when they are exposed to the terrorist groups again. The mathematical analysis of the model shows a graph of the backward bifurcation phenomenon for certain values of the parameters. Using parameter values associated with the terrorism situation, we showed how the re-cycling of terrorists can trigger the backward bifurcation in our model. A backward bifurcation will make it a lot more difficult to control terrorists' activities because a critical value of the reproduction number R_{ter}^1 (see Figure 3) which is much less than 1 is needed to achieve a terrorists-free state.

The basic reproduction number R_{ter} for the model is computed using the next-generation matrix method. However, re-joining the terrorist group creates the possibility of a backward bifurcation as shown in Figure 3. Therefore, controlling the basic reproduction number R_{ter} (i.e., controlling contact rate and recovery rate) alone will not be sufficient to control the terrorist's activities. We should therefore control the possibility of re-joining the terrorists group to eradicate the terrorist's activities.

When the value of the scaling factor of the re-cycling parameter is increased from 0 to 1 as shown in Figures 5–9, the bifurcation changes from backward bifurcation to forward bifurcation. This is because the recycling rate is $((1-\delta)\beta = H)$. Therefore, $(\frac{\partial \delta}{\partial H} = -\frac{1}{\beta})$. Hence the scaling factor is a decreasing function of the recycling rate. So, increasing the scaling factor will reduce the re-cycling rate while reducing the scaling factor will increase the re-cycling rate. Figures 10–14 show the time series plots of the scaling factor of re-cycling terrorist (δ), the contact rate (β), terrorism rate (ξ), the imprisonment rate (σ), and the recovery rate (γ). Figure 10 shows that increasing the scaling factor from 0 to 1 or decreasing the recycling rate from 1 to 0 will reduce the population of the terrorists' class over time until the curve flattens. Figure 11 shows that increasing the contact rate from 0.0001 to 0.1 will increase the population of the terrorists' class as time over time. Figure 12 shows that increasing the rate at which exposed individuals join the terrorists' group from 0.0001 to 0.1 will increase the terrorists' class over time. Hence, the government should try to prevent our youths from being exposed to terrorism to avoid indoctrination. Figure 13 shows that increasing the imprisonment rate from 0.0001 to 0.7 will reduce the terrorist population over time. Therefore, the authorities should ensure that all terrorism-related cases offenders are sent to prison without bail to serve as a deterrent to anyone who is thinking of joining this deadly group. Figure 14 shows that increasing the rate of repentance from 0.01 to 0.9 will reduce the population of terrorists. The authorities should therefore rehabilitate those individuals who are involved in the act of terrorism and are caught to bring awareness to them on the negative impact of terrorism in the society.

6. Conclusion

In conclusion, our mathematical model highlights the critical role of re-cycling and various parameters in driving terrorist activities. We have shown that controlling the basic reproduction number alone is insufficient to combat terrorism. By managing re-cycling rates and implementing effective policies such as imprisonment, rehabilitation, and preventing exposure, we can make significant strides in reducing the terrorist population. Our findings emphasize the multifaceted approach required for counterterrorism efforts.

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